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Longevity gap and public pensions: a minimal model

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ABSTRACT

The strong and increasing positive correlation between lifetime income and life expectancy (the longevity gap) has recently been widely studied. In this paper we employ the simplest, minimal model to demonstrate the impact of this long-neglected fact on the various types of public pension systems, especially on the issue of progressivity and neutrality.

JEL codes: D10, H55, I38

Keywords: social security, public finance, income-dependent life expectation, longevity gap

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Élettartam-rés és tb-nyugdíjrendszer

SIMONOVITS ANDRÁS

ÖSSZEFOGLALÓ

[Az életpálya-jövedelem és a várható élettartam közti erős és időben növekvő pozitív korreláció (az élettartam-rés) egyre inkább az érdeklődés központjába került. Cikkemben a legegyszerűbb, minimális modellt alkalmazom, hogy bizonyítsam: e sokáig elhanyagolt tény hat a különféle tb-nyugdíjrendszerekre, különösen a nyugdíj degresszivitására és semlegességére.

JEL: D10, H55, I38

Kulcsszavak: társadalombiztosítás (tb), jövedelemfüggő várható élettartam, élettartam-rés

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July 4, 2021

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Abstract

The strong and increasing positive correlation between lifetime income and life expectancy (the longevity gap) has recently been widely studied. In this paper we employ the simplest, minimal model to demonstrate the impact of this long-neglected fact on the various types of public pension systems, especially on the issue of progressivity and neutrality.

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1 Introduction

Considering public pension systems, there have always been a lively debate on the optimal compromise between efficiency and solidarity. For example, efficiency (i.e. sufficient labor supply) requires that the link between contributions and pension benefits be as strong as possible, while solidarity (ensuring that even low income pensioners have adequate income) requires just the opposite. Using the classical assumption of unisex social insurance, namely that every member of a given cohort has the same life expectancy, one can design simple attractive benefit rules. This problem becomes much more complex if the classical assumption is replaced by a more realistic one: the higher the lifetime income, the longer the life expectancy. The difference between highest and lowest income levels' life expectancies are frequently called *longevity gap*. (Note that Bravo et al. (2021) discuss *life expectancy gap*, defined as the difference between cohort and period life expectancies.)

Already Liebmann (2002) exposed this longevity gap in the US and drew the conclusion: the apparently very strong progressivity of the US Social Security, namely that with the rise of the average indexed monthly earning, smaller and smaller parts are taken into account in the primary insurance amount, losing force. In fact, considering lifetime net contribution balances, redistribution is much smaller than implied by the progression. Along similar lines, Whitehouse and Zaidi (2008) drew implications for pension policy from socioeconomic differences in mortality. Furthermore, Breyer and Hupfeld (2009a, b) questioned the fairness of early retirement provisions and of the strong link between earnings and benefits, respectively. In a calibrated dynamic general equilibrium model, Fehr et al. (2013) determined the optimal progressivity of the German public pension system.

Some years ago, starting a new wave, a lot of researchers documented the strong and increasing positive correlation between lifetime income and life expectancy (e.g. National Academies of Sciences, Engineering, and Medicine, 2015; Chetty et al., 2016). We shall only cite two US data-sets: (i) The gap between the male life expectancies at 65 of the richest and the poorest 1 percentiles was 14.6 years in 2014; (ii) Between 2001 and 2014, the male life expectancy of the richest 5 percentile rose by 2.34 years and the poorest 5 percentiles' only rose by 0.04 years. As a result of this new recognition, economists have paid much more attention to the longevity gap than before. To name only few new studies: Pestieau and Ponthiere (2016) published a survey on the longevity variation and the welfare state, emphasizing historical processes and many theoretical complexities. Sánchez et al. (2017) reexamined the redistributive effects of the US Social Security system and Ayuso et al. (2017) addressed the longevity heterogeneity in pension design. Most recently, Haan et al. (2020) compared the difference of life expectancies at age 65 between the highest and the lowest deciles of West German males, and found that the gap rose from 4 years (30%) born 1926-28 to 7 years (50%) born in 1947-49. They also extended the discussion to the spouses' incomes and life expectancies.

Sheshinski and Caliendo (2020) developed a relatively simple theoretical model, relying on the parameters of the US Social Security system and the increasing longevity gap. They introduced a simple measure of redistribution, and showed that between 1930 and 1960, it decreased to 1/4 of the original. They developed alternatives for preserving the original redistribution at differential tax increase or benefit reduction and studied the welfare properties of these reforms.

In the World Bank's recent twin volumes (Holzmann et al., eds, 2020), three chapters discussed the impact of increasing longevity gap on pension schemes. Lee and Sánchez-

Romero (2020); Palmer and Zhao de Gosson de Varennes (2020) and Holzmann et al. (2020) came up with reform proposals on NDC. Since our paper is closely related to the third study, we discuss it more thoroughly. Holzmann et al. (2020) added English and Welsh data to Chetty at al. (2016)'s US data. Considering five designs with disaggregated annual contributions and benefits, they calculated the designs' income-dependent implicit tax/subsidy rates due to the income-dependence of life expectancy. It is of special interest that they determined the linear combination of proportional and basic benefits which *minimizes* the variance of the pensions [(14.10–11)].

In the present paper, we develop an even simpler model than theirs: in our *minimal* model, workers are employed for a unitary period for an age-invariant gross wage, pay pension contributions; when retire, receive a *linear combination* of the proportional and the basic (or flat) benefits for a shorter period. (Note that more complex, piecewise linear, concave benefit rules are frequently employed, but our linear version is a good approximation (Disney, 2004).) Considering a quite general life expectancy–wage-schedule, we study individual lifetime balances rather than implicit taxes/subsidies. Note that we replaced incomes by wages, to avoid the income $\rightarrow$ life expectancy $\rightarrow$ old-age income circularity, though in addition to wages, old-age income also influences life expectancy (Philipson and Becker, 1998). Like others, we were unable to make the duration depend on net rather than gross wages. We introduced personal income taxes (paid by workers) and basic income (received by workers as well as pensioners) to shift part of the burden of income redistribution from the pension system. To make room for variable pension contribution rates, we introduced private savings and young- and old-age consumption as well. Introducing lifetime utility functions, we could also define a social welfare function as expected utility.

The main results are as follows: (a) In the basic scheme, the existence of the longevity gap reduces redistribution from the high earners to the low earners. (b) In the proportional pension scheme, the longevity gap introduces redistribution into the system but only the highest earners gain. (c) If the contribution rate is given and the government aims to minimize lifetime pension redistribution, measured by the standard deviation of lifetime balances, then the *minimizer* proportional share can be explicitly determined. (d) To punish pension redistribution, the social welfare function is modified by deducting the product of standard deviation of lifetime balances and of the fine rate from the expected utility (as in Eső et al., 2011). Increasing the fine rate, the optimal contribution rate is first declining then climbs back to toward the value of Example 2, the optimal proportionality share is increasing, converging to the minimizer. We have extensively relied on numerical illustrations using Chetty at al. (2016) and the convergence is reached already for the unit fine rate.

We have tried to consider the simplest issues and left out equally important but more complicated problems. Just to name three omissions: (i) Indexation to wages rather than to prices favors longer-lived pensioners (from Legros, 2006 to Simonovits, 2018, Section 14.4). (ii) The usual delayed retirement credit overlooks that workers delaying retirement are not only more diligent but also healthier and might live longer than others (starting with Diamond, 2003; Eső and Simonovits, 2002). (iii) The transfers may diminish work participation and taxation (cf. Prescott (2004) for short-term transfers).

The structure of the remainder of the paper is as follows. Section 2 presents the basic model. Section 3 investigates the approximation of neutrality. Section 4 analyzes the socially optimal mixed system. Section 5 concludes. An Appendix analyzes the impact of eliminating the longevity gap and the indeterminacy of the social optimum.

2 Basic model

In this Section we introduce our basic model. Assume that every worker is characterized by her total wage compensation w, she pays pension contribution τw for a period of unit length and receives a per-period benefit $b(w)$ for a period of length $m(w)$, $0 < m(w) < 1$, where $b(\cdot)$ and $m(\cdot)$ are weakly and strictly increasing functions, respectively. Considering only public pensions, we assume unisex populations! Obviously, $m(w)$ is the conditional life expectancy at retirement at earning w. Since we neglect risk (see e.g. Fleurbaey et al., 2016), we shall speak of *duration* of retirement. We shall normalize average wage to 1, and denote the average duration by $\mu = \mathbf{E}m(w)$. To avoid absurd outcomes, we shall make the following additional realistic assumption: $m(\cdot)$ is (strictly) concave. By Jensen inequality, typically $\mu < m(1)$. As a consequence, the ratio of duration to wage $[m(w)/w]$ is a decreasing function of wage. We assume a general cumulated distribution function $F(w)$ and denote expected values by operator $\mathbf{E}$.

To measure (expected) lifetime redistribution, we shall consider
The *lifetime balance* of a worker with wage w and benefit $b(w)$:

$$z(w, b(w)) = \tau w - b(w)m(w). \tag{3}$$

We call a pension system *neutral* if the corresponding lifetime balance is identically zero:

$$z(w, b^{\mathrm{N}}) \equiv 0. \tag{3N}$$

(3N) yields the neutral benefit rule

$$b^{\mathrm{N}}(w) = \frac{\tau w}{m(w)}. \tag{4N}$$

This scheme has a number of problems: (i) Income is not the only determinant of life expectancy. (ii) The relation may change over time. (iii) That design may not be politically sustainable given the woeful lack of financial literacy. It will only serve us as a useful zero point.

Though the system is generally not neutral, we assume that the pension system is *balanced*, i.e. the *expected* (value of the lifetime) balance is equal to zero. Using (3) and $\mathbf{E}w = 1$:

$$\mathbf{E}z(w, b(w)) = \tau - \mathbf{E}[b(w)m(w)] = 0. \tag{5}$$

The simplest scheme, called *basic benefit* (B), neglects any proportionality to contributions and pays the same benefit to everybody:

$$b^{\mathrm{B}}(w) = b.$$

Substituting it into (5) yields

$$b^{\mathrm{B}}(w) = \frac{\tau}{\mu} \tag{4B}$$

and the resulting wage-dependent lifetime balance is given by

$$z(w, b^{\mathrm{B}}) = \tau w - \frac{\tau}{\mu} m(w). \tag{3B}$$

We shall define w_{B} as that wage for which $z(w_{\mathrm{B}}, b^{\mathrm{B}}) = 0$, i.e. $\mu w_{\mathrm{B}} = m(w_{\mathrm{B}})$. Anticipating our empirical data (cf. Table 1), we shall assume that $w_{\mathrm{B}} \geq 1$. Taking into account our assumption on the decline of $m(w)/w$, we have proved

Theorem 1. *In the basic benefit scheme (4B), the workers earning above w_{B} are the net contributors (losers): $z(w, b^{\mathrm{B}}) > 0$ if and only if $w > w_{\mathrm{B}}$.*

Example 1. We shall occasionally use the following analytical formula:

$$m(w) = m_1 w^a, \qquad 0 < a < 1 \qquad \text{and} \qquad m_1 = m(1).$$

Then $m(w_{\mathrm{B}})/w_{\mathrm{B}} = \mu$ reduces to $m_1 w_{\mathrm{B}}^{a-1} = \mu$, i.e. $w_{\mathrm{B}} = (m_1/\mu)^{1/(1-a)}$. Since $m_1 > \mu$, $w_{\mathrm{B}} > 1$.

In a third type of public pension systems, the benefit is *proportional* to contributions or equivalently, to wages:

$$b^{\mathrm{P}}(w) = \beta w, \qquad \beta > 0.$$

Now the balance condition (5) reduces to $\tau = \beta \mathbf{E}(wm(w))$. To highlight the role of $\mathbf{E}(wm(w))$, we introduce notation

$$\rho = \frac{\mathbf{E}(wm(w))}{\mu}.$$

Here ρ is equal to 1 + the relative covariance of wage w and duration $m(w)$, where $\mathbf{E}m(w) = \mu$ and $\mathbf{E}w = 1$. Then the proportional benefit is given by

$$b^{\mathrm{P}}(w) = \frac{\tau w}{\mu \rho} \tag{4P}$$

and the corresponding balance is equal to

$$z(w, b^{\mathrm{P}}(w)) = \tau w - \frac{\tau w}{\mu \rho} m(w). \tag{3P}$$

Due to the generically positive correlation of $m(w)$ and w, $\rho > 1$ holds. The naive calculation neglects the deviation of ρ from 1, but this distorts the calculations.

For any continuous and strictly increasing function $m(\cdot)$, there exists a wage w_{P}, where the proportional lifetime balance is zero. (3P) yields the implicit equation $m(w_{\mathrm{P}}) = \rho\mu$ for w_{P}. Now we have

Theorem 2. *Because $\rho > 1$, in the proportional scheme (4P), workers earning below wage w_{P} are the net contributors (losers): $z^{\mathrm{P}}(w) > 0$ for $w < w_{\mathrm{P}}$, others are the net beneficiaries.*

Example 1. (continued.) For $m(w) = m_1 w^a$, $m(w_{\mathrm{P}}) = \rho\mu$ reduces to $m_1 w_{\mathrm{P}}^a = \rho\mu$, i.e. $w_{\mathrm{P}} = (\rho\mu/m_1)^{1/a}$.

Remark. It is of interest that the basic benefit is greater than the proportional benefit at the average wage: $b^{\mathrm{B}}(1) > b^{\mathrm{P}}(1)$. It is an open question if $w_{\mathrm{P}} > w_{\mathrm{B}}$ or not.

In our numerical illustrations we shall use the average *decile* incomes and durations taken by unisex aggregation from Chetty et al. (2016), and the corresponding duration-to-wage-ratio (Table 1). It can be seen that the latter is steeply declining with the wage. The average wage lies close to $w_8 = 1.06$ and the corresponding duration $m_8 \approx 0.5$ is higher than the average duration $\mu = 0.45$. We also present the proportional and the basic systems's balances. Note that in the former, only the highest decile practically gains, while in the latter, the decile 9's loss is also sizable. Despite the gap, the absolute values of the proportional balances are much less than that of the basic ones.

Table 1: Decile distributions

Decile index i	Relative wage w_i	Relative duration m_i	Duration-to-wage-ratio m_i/w_i	P-balance z_i^{P}	B-balance z_i^{B}
1	0.075	0.333	4.433	0.005	–0.132
2	0.205	0.370	1.805	0.011	–0.123
3	0.315	0.398	1.262	0.014	–0.113
4	0.428	0.425	0.993	0.015	–0.103
5	0.552	0.448	0.811	0.014	–0.088
6	0.692	0.466	0.673	0.012	–0.068
7	0.856	0.484	0.565	0.010	–0.043
8	1.064	0.504	0.473	0.004	–0.011
9	1.404	0.525	0.374	–0.007	0.048
10	4.409	0.558	0.126	–0.078	0.634

As was already mentioned in the Introduction, a lot of progressive schemes can be approximated by a convex linear combination of the proportional and the basic benefits. We consider the *mix* (M) of schemes P and B, with nonnegative shares $\alpha, 1-\alpha$, $0 \le \alpha \le 1$. We shall make use of the fact that the benefit is still proportional to the contribution rate:

$$b^{\mathrm{M}}(w) = \frac{\tau B(\alpha, w)}{\mu}, \qquad \text{where} \qquad B(\alpha, w) = \rho^{-1}\alpha w + 1 - \alpha. \tag{4M}$$

Substituting (4M) into (3) results in

$$z(w, b^{\mathrm{M}}) = \alpha z(w, b^{\mathrm{P}}) + (1-\alpha) z(w, b^{\mathrm{B}}) = \tau w - \frac{\tau(\rho^{-1}\alpha w + 1 - \alpha)}{\mu} m(w) \tag{3M}$$

preserving (5).

Following Holzmann et al. (2020), we define the wage-dependent *implicit tax/subsidy rate* $\varphi(w)$ as the ratio of the lifetime balance to the wage:

$$\varphi(w) = \frac{z(w, b^{\mathrm{M}})}{w}. \tag{6}$$

Substituting (3M) into (6) results in

$$\varphi(w) = \tau - \frac{\tau[\rho^{-1}\alpha + (1-\alpha)/w]}{\mu} m(w).$$

The dependence of $\varphi(w)$ on α and $m(\cdot)$ is much more complex than in the case of balances.

3 Toward neutrality

Until now we have taken the macro parameter values τ, α as given. Let us now turn to a systematic investigation of the impact of these pairs of parameters on the outcome. We start with the question of neutrality, i.e. the universal zero lifetime balance.

In general, neutrality cannot be achieved with mixing P and B but it can be approximated by *minimizing the variance* of the lifetime balances

$$\sigma_z^2(\alpha) = \mathbf{E}[z^{\mathrm{M}}]^2.$$

Note that this measure is symmetric, i.e. it does not distinguish between the direction of redistribution: whether the low earners support the high earners (as with P) or the high earners support the low earners (as with B).

Starting with (3P) and (3B), the two extreme variances are respectively equal to

$$\mathbf{E}z^{\mathrm{P}^2} = \tau^2\mathbf{E}[w - \rho^{-1}\mu^{-1}wm(w)]^2 \qquad \text{and} \qquad \mathbf{E}z^{\mathrm{B}^2} = \tau^2\mathbf{E}[w - \mu^{-1}m(w)]^2.$$

To get an intuition on their order, it is worth considering two extreme cases. (i) In the classic case of $m(w) \equiv \mu$ (and $\mathbf{E}w = 1$), $\mathbf{E}z^{\mathrm{P}^2} = 0 < \tau^2\mathbf{E}(w-1)^2 = \mathbf{E}z^{\mathrm{B}^2}$ is valid. (ii) If $m(w) = \mu w$ (the excluded linear case), then the basic benefit is identical to the 'neutral' one [(4N)], therefore the opposite holds: $\mathbf{E}z^{\mathrm{P}^2} > 0 = \mathbf{E}z^{\mathrm{B}^2}$. Without defining the meaning of 'realistic', we only risk

Conjecture 1. *For realistic duration–wage schedules, the variance of the proportional scheme's balance is lower than the basic benefit's:*

$$\mathbf{E}z^{\mathrm{P}^2} < \mathbf{E}z^{\mathrm{B}^2}.$$

We continue with the mixed benefits [(4M)] and prove a theorem on the proportionality share minimizing the variance of the balances.

Theorem 3. *For any given contribution rate τ, the minimal variance of the lifetime balances in the mixed system is attained at the proportionality share*

$$\alpha^* = \frac{\mathbf{E}[(z^{\mathrm{B}} - z^{\mathrm{P}})z^{\mathrm{B}}]}{\mathbf{E}[z^{\mathrm{P}} - z^{\mathrm{B}}]^2} \tag{8}$$

where z^{P} and z^{B} are given in (3P) and (3B), respectively.

Remarks. 1. Since z^{P} and z^{B} are proportional to τ, the minimizer α^* is independent of τ.

2. It is not easy to prove that the minimizer share lies between 0 and 1 but we shall argue for its validity later on. For the data used in Table 1, $\alpha^* = 0.892$ and the corresponding standard deviation is almost zero: $\sigma_{z^*} = 0.004$.

Proof. Relying on (4M),

$$\mathbf{E}z^{\mathrm{M}^2} = \mathbf{E}[\alpha(z^{\mathrm{P}} - z^{\mathrm{B}}) + z^{\mathrm{B}}]^2.$$

Using the additivity of the expectation, we have the quadratic function

$$\sigma^2(\alpha) = \mathbf{E}[z^{\mathrm{P}} - z^{\mathrm{B}}]^2\alpha^2 + 2\mathbf{E}[(z^{\mathrm{P}} - z^{\mathrm{B}})z^{\mathrm{B}}]\alpha + \mathbf{E}[z^{\mathrm{B}}]^2.$$

The minimum of $\sigma^2(\alpha)$ is attained at (8). □.

Note that if $\mathbf{E}z^{\mathrm{P}^2} < \mathbf{E}z^{\mathrm{B}^2}$ (Conjecture 1), then $\alpha^* \ge 0$.

4 Social welfare maximization

Having studied the approximation of neutrality, we turn now to social welfare maximization. We have to introduce consumption pairs, utility and social welfare functions. We shall also introduce a basic income to diminish the burden of redistribution on the pension system. In a proportional personal income tax system, worker earning w pays tax θw to finance a basic income ι for workers and $\delta^*\iota$ for pensioners, where $\delta^* \in (0,1]$ accounts for lower family size of the pensioners. Obviously, $(1+\mu\delta^*)\iota = \theta$.

Without private savings, the consumption pairs would be equal to

$$c_0(w) = (1-\tau-\theta)w + \iota \qquad \text{and} \qquad d_0(w) = \tau\mu^{-1}B(\alpha, w) + \delta^*\iota.$$

We also add *private savings* to ensure adequate old-age consumption for the high-earners if the pension is too progressive or for everybody if the contribution rate is too low. Then the consumption functions are modified as

$$c(w) = c_0(w) - s(w) \qquad \text{and} \qquad d(w) = d_0(w) + m(w)^{-1}Rs(w), \tag{9}$$

where R stands for the cumulative interest factor. (It is heroically assumed that every saver can buy a perfect private life annuity matching her life expectancy $m(w)$ without paying any extra fee.)

We shall use a very simple lifetime utility function with a discount factor $\delta \in (0,1]$:

$$U(w, c(w), d(w)) = \log c(w) + \delta m(w) \log d(w). \tag{10}$$

Turning to the social welfare function, we adopt a method introduced by Feldstein (1985): the government uses a weaker discount, denoted by the same factor δ^* as before, $\delta \le \delta^* \le 1$. (If $\delta^* = 1$, then the varying family size would be neglected.) We do not follow, however, Feldstein's undervaluation of the public pension system (see Simonovits, 2018, Section 3.2 and Appendix D). One reason for introducing mandatory public pensions is to force shortsighted workers to save more than they would voluntarily.

Example 2. Having a representative agent, with $w = 1$, $\alpha = 1$, $\theta = 0$ and no saving, the social welfare function is given by

$$V[\tau] = \log(1-\tau) + \mu\delta^* \log(\mu\tau).$$

Taking its derivative and setting the derivative to zero, the optimal contribution rate can be determined:

$$0 = V'[\tau] = \frac{1}{1-\tau} + \frac{\mu\delta^*}{\tau} \Rightarrow \tau^* = \frac{\mu\delta^*}{1+\mu\delta^*}.$$

The corresponding consumption pair are

$$c^* = \frac{1}{1+\mu\delta^*} \qquad \text{and} \qquad d^* = \frac{\delta^*}{1+\mu\delta^*}. \tag{11}$$

Numerically, $\delta^* = 0.5$ and $\mu = 0.45$ yields $\tau^* = 0.184$, $c^* = 0.816$ and $d^* = 0.368$.

Returning to the general case, we shall now determine the optimal private savings. Insert (9) into (10):

$$U[\delta, s(w)] = \log(c_0 - s(w)) + \delta m(w) \log(d_0(w) + m(w)^{-1}Rs(w)). \tag{12}$$

Take the partial derivative of (12):

$$U'_s[\delta, s] = \frac{-1}{c_0(w) - s} + \frac{\delta}{d_0(w)R^{-1} + m(w)^{-1}s} = 0$$

and after rearrangement:

$$d_0(w)R^{-1} + m(w)^{-1}s = \delta c_0(w) - \delta s.$$

Solving for s and replacing the possible negative solution by 0 yields

$$s^{\mathrm{o}}(w) = \frac{[\delta c_0(w) - d_0(w)R^{-1}]_+}{\delta + m(w)^{-1}},$$

where x_+ is the positive part of a real x. Substituting $s^{\mathrm{o}}(w)$ into (12) yields the indirect utility function $U^{\mathrm{o}}[\delta, w] = U[\delta, s(w)]$.

Consider first the expected utility function:

$$V(\tau, \theta, \alpha) = \mathbf{E}U^{\mathrm{o}}[\delta^*, w]. \tag{13}$$

If there were no efficiency constraints, then the social optimum could be achieved by an extremely simple method; no pension system: $\tau = 0$, and total age-specific income redistribution: $\theta^* = 1$ with $\iota^* = 1/(1 + \mu\delta^*)$. Then regardless of the original, pre-tax wages, everybody would have the consumption pair (11) of Example 2.

Leaving behind this utopia, we fix $\theta = 0.1$ and $\delta^* = 0.5$, $\delta = 0.25$ and $R = 1$. (Obviously, the value of the cumulated interest factor plays a decisive role but here we confine our attention to the redistribution within the public pension system.) First we present two related paths in Tables 2 and 3. Note that $\tau(1) < \tau(2)$, $\alpha(1) < \alpha(2)$, $b_1(2) < b_1(1)$ and $b_{10}(2) > b_{10}(1)$. Furthermore, among the twenty savings, only $s_{10}(1) = 0.122$ is positive.

Table 2: Paths with low contribution rate and share: $\tau = 0.13$ and $\alpha = 0.35$

Wage w	Pension b	Young consumption c	Old consumption d	Balance z
0.075	0.194	0.139	0.235	–0.055
0.205	0.206	0.239	0.246	–0.049
0.315	0.215	0.324	0.256	–0.045
0.428	0.225	0.411	0.266	–0.040
0.552	0.236	0.507	0.277	–0.034
0.692	0.249	0.615	0.290	–0.026
0.856	0.263	0.741	0.304	–0.016
1.064	0.282	0.901	0.323	–0.004
1.404	0.312	1.163	0.353	0.019
4.409	0.579	3.354	0.839	0.250

Table 3: Paths with high contribution rate and share: $\tau = 0.16$ and $\alpha = 0.6$

Wage w	Pension b	Young consumption c	Old consumption d	Balance z
0.075	0.156	0.137	0.197	–0.040
0.205	0.180	0.233	0.221	–0.034
0.315	0.201	0.315	0.242	–0.029
0.428	0.222	0.398	0.263	–0.026
0.552	0.245	0.490	0.286	–0.021
0.692	0.272	0.594	0.313	–0.016
0.856	0.302	0.715	0.343	–0.009
1.064	0.341	0.869	0.382	–0.002
1.404	0.405	1.121	0.446	0.012
4.409	0.968	3.344	1.009	0.166

Turning to social welfare maximization, we shall deduct a *fine* for pension redistribution (proportional to the standard deviation of balances) from the expected utility function:

$$V(\tau, \theta, \alpha) = \mathbf{E}U^{\circ}[\delta^*, w] - \zeta\sigma_z, \qquad \zeta > 0. \tag{14}$$

Note that as the fine rate ζ rises, our maximization problem converges to the minimization of the standard deviation of balances. Finally, to make the private savings and the public pension contributions comparable, we calculate $S = \mathbf{E}s$.

To start with, Table 4 just maps the social welfare function in a very rough way: we pick up two contribution rates 0.13 and 0.16; five equally distributed shares between 0 and 1 and three fine rates 0, 0.5 and 1, respectively. Note that the values of Vs are only comparable within a column but the differences have no direct economic meaning. One and two stars denote weak and strong optima, respectively. Note that especially for the medium fine rate, there are a lot of optima—quite isolated from each other. In particular, the strongly differing paths displayed in Tables 2 and 3 are both socially optimal for $\zeta = 0.474$ (see Table 4), quite close to 0.5.

Table 4: Rough mapping

Contribution rate	Proportionality share	Standard deviation	Saving	Social welfare for fine rates		
τ	α	σ_z	S	$\zeta_1 = 0$	$\zeta_2 = 0.5$	$\zeta_3 = 1.0$
0.13	0.00	0.142	0.026	–0.792*	–0.863	–0.934
	0.25	0.102	0.016	–0.808	–0.859*	–0.910
	0.50	0.062	0.006	–0.827	–0.858*	–0.889
	0.75	0.023	0.000	–0.852	–0.863	–0.874*
	1.00	0.017	0.000	–0.888	–0.897	–0.906
0.16	0.00	0.174	0.022	–0.787**	–0.875	–0.962
	0.25	0.126	0.009	–0.802	–0.865	–0.928
	0.50	0.077	0.000	–0.820	–0.859*	–0.897
	0.75	0.028	0.000	–0.844	–0.858*	–0.872**
	1.00	0.021	0.000	–0.883	–0.894	–0.905

Table 5 deepens the analysis of the previous table. We display the dependence of optimal characteristics on the fine rate for redistribution $\zeta \in [0,1]$. To follow a sensitive change, we apply finer resolution in $\zeta \in (0.4, 0.6)$ than otherwise. The outcomes are as follows. As the fine rises from 0 to 0.4, the optimal contribution rate sinks from 0.17 to 0.11 keeping out any proportional benefits. Between 0.4 and 1, the optimal contribution rate rises to 0.17 and the optimal proportionality share rises from 0 to 0.85 – a proxy for the minimizer.

Table 5: The optimal contribution and proportionality as a function of the fine rate: gap

Fine rate for redistribution ζ	Optimal contribution rate τ^{o}	Optimal proportionality share α^{o}	Standard deviation of balances $\sigma_z(\nu, \alpha^{o}(\nu, \zeta))$	Savings S
0.0	0.17	0.0	0.185	0.020
0.1	0.15	0.0	0.163	0.023
0.2	0.14	0.0	0.153	0.025
0.3	0.13	0.0	0.142	0.026
0.4	0.11	0.0	0.120	0.030
0.42	0.13	0.2	0.110	0.018
0.44	0.13	0.3	0.094	0.014
0.46	0.13	0.3	0.094	0.014
0.48	0.16	0.6	0.057	0
0.50	0.16	0.6	0.057	0
0.52	0.16	0.7	0.038	0
0.60	0.16	0.75	0.028	0
0.65	0.16	0.80	0.019	0
0.70	0.16	0.85	0.009	0
0.75	0.16	0.85	0.009	0
0.80	0.16	0.85	0.009	0

5 Conclusions

We have completed our paper. Creating a minimal model of the longevity gap and the public pension system, we were able to revisit a number of interesting and important issues in pension design. What is the optimal combination of proportional and basic pensions when the expected time spent in retirement grows with lifetime earnings? How much can the neutrality of the pension system be approximated? We have received sharp results: separation of the gainers and losers; determination of the socially optimal contribution rate and the optimal proportionality share but at the cost of neglecting a lot of important factors. To name just a few neglected factors: (a) rising life expectancy and length of contributions, (b) rising real wages and the indexation; (c) endogenous labor supply. Taking them into account, however, is beyond the scope of the present paper.

6 Appendix. No longevity gap

In this Appendix, we discuss what happens if the longevity gap is eliminated.

Table A1 recalculates Table 5 having eliminated the longevity gap. The main impact is ad follows: for fine rate 1.2, the proportional system is the socially optimal one. Below $\zeta = 1.2$, however, some redistribution serves for intentional redistribution.

Table A1. The optimal proportionality share and the standard deviation as a function of the fine: no gap

Fine rate for redistribution ζ	Optimal contribution rate τ^o	Optimal proportionality share α^o	Standard deviation of balances $\sigma_z(\nu, \alpha^o(\nu, \zeta))$	Savings S
0.0	0.17	0.0	0.204	0.017
0.2	0.14	0.0	0.168	0.021
0.4	0.12	0.0	0.144	0.024
0.42	0.11	0.0	0.132	0.026
0.44	0.12	0.1	0.130	0.020
0.46	0.12	0.2	0.115	0.016
0.48	0.12	0.2	0.115	0.016
0.50	0.13	0.3	0.109	0.011
0.52	0.13	0.4	0.094	0.007
0.54	0.15	0.6	0.072	0.000
0.6	0.15	0.65	0.063	0
0.7	0.15	0.75	0.045	0
0.8	0.15	0.80	0.036	0
0.9	0.16	0.90	0.019	0
1.0	0.16	0.90	0.019	0
1.1	0.16	0.95	0.010	0
1.2	0.17	1.00	0.000	0

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