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The stable marriage problem with ties and restricted edges

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ABSTRACT

In the stable marriage problem, a set of men and a set of women are given, each of whom has a strictly ordered preference list over the acceptable agents in the opposite class. A matching is called stable if it is not blocked by any pair of agents, who mutually prefer each other to their respective partner. Ties in the preferences allow for three different definitions for a stable matching: weak, strong and super-stability. Besides this, acceptable pairs in the instance can be restricted in their ability of blocking a matching or being part of it, which again generates three categories of restrictions on acceptable pairs. Forced pairs must be in a stable matching, forbidden pairs must not appear in it, and lastly, free pairs cannot block any matching.

Our computational complexity study targets the existence of a stable solution for each of the three stability definitions, in the presence of each of the three types of restricted pairs. We solve all cases that were still open. As a byproduct, we also derive that the maximum size weakly stable matching problem is hard even in very dense graphs, which may be of independent interest.

JEL codes: C63, C78

Keywords: stable matchings, restricted edges, complexity

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A stabil párosítás probléma gyengén rendezett listákkal és korlátozott élekkel

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ÖSSZEFOGLALÓ

A klasszikus stabil párosítás problémában nők és férfiak egy halmaza adott. Mindenki felállít egy szigorúan rendezett preferencialistát az ellenkező nem néhány tagjáról. Akkor nevezünk egy párosítást stabilnak, ha azt nem blokkolja egyetlen pár sem. Egy pár akkor blokkolja a párosítást, ha mindkét tagja magasabban rangsorolja egymást, mint a párosításban hozzájuk rendelt személyeket. Ha a preferencialisták nem szigorúan, hanem gyengén rendezettek, akkor háromféle stabilitási definícióval dolgozhatunk: gyenge, erős és szuper-stabilitással. Egyes párok lehetnek korlátozottak is: a kötelező párokat muszáj, a tiltott párokat pedig tilos tartalmaznia a keresett stabil párosításnak. A szabad élek lehetnek párosítás tagjai, de nem blokkolhatnak párosítást.

Cikkünkben bonyolultságelméleti szempontból tanulmányozzuk a stabil megoldás létezését mindhárom stabilitási definícióra, mindhárom féle korlátozott él jelenlétében. Minden eddig nyitott kérdést megválaszolunk. Az is kiderül, hogy a maximális méretű gyengén stabil párosítás problémája még nagyon sűrű gráfokban is NP-teljes.

JEL: C63, C78
Kulcsszavak: stabil párosítás, korlátozott élek, bonyolultság

The stable marriage problem with ties and restricted edges

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Abstract

In the stable marriage problem, a set of men and a set of women are given, each of whom has a strictly ordered preference list over the acceptable agents in the opposite class. A matching is called stable if it is not blocked by any pair of agents, who mutually prefer each other to their respective partner. Ties in the preferences allow for three different definitions for a stable matching: weak, strong and super-stability. Besides this, acceptable pairs in the instance can be restricted in their ability of blocking a matching or being part of it, which again generates three categories of restrictions on acceptable pairs. Forced pairs must be in a stable matching, forbidden pairs must not appear in it, and lastly, free pairs cannot block any matching.

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1. Introduction

In the classical *stable marriage problem* (SM) [14], a bipartite graph is given, where one side symbolizes a set of men U, while the other side symbolizes a set of women W. Man u and woman w are connected by the edge uw if they find one another mutually acceptable. In the most basic setting, each participant provides a strictly ordered preference list of the acceptable agents of the opposite gender. An edge uw *blocks* matching M if it is not in M, but each of u and w is either unmatched or prefers the other to their respective partner in M. A *stable matching* is a matching not blocked by any edge. From the seminal paper of Gale and Shapley [14], we know that the existence of such a stable solution is guaranteed and a stable matching can be found in linear time.

Several real-world applications [6] require a relaxation of the strict order to weak order, or, in other words, preference lists with ties, leading to the *stable marriage problem with ties* (SMT) [16, 18, 24]. When ties occur, the definition of a blocking edge needs to be revisited. In the literature, three intuitive definitions are used, namely weakly, strongly and super-stable matchings [16]. According to weak stability, a matching is *weakly blocked* by an edge uw if agents u and w both strictly prefer one another to their partners in the matching. A *strongly blocking* edge is preferred strictly by one end vertex, whereas it is not strictly worse than the matching edge at the other end vertex. A *super-blocking edge* is at least as good as the matching edge for both end vertices in the super-stable case. Super-stable matchings are strongly stable and strongly stable matchings are weakly stable by definition, because weakly blocking edges are strongly blocking, and strongly blocking edges are super-blocking at the same time.

Weak and strong stability serve as the goal to achieve in most applications, such as college admission programs. In most countries, colleges are not required to rank all applicants in a strict order of preference, hence large ties occur in their lists. According to the equal treatment policy used in Chile and Hungary for example, it may not occur that a student is rejected from a college preferred by her, even though other students with the same score are admitted [7, 29]. Other countries, such as Ireland [9], break ties with lottery, which gives way to a weakly stable solution according to the original, weak order. Super-stable matchings can represent safe solutions if agents provide

	uw must be in M	uw can be in M	uw must not be in M
uw can block M	**forced**	unrestricted	**forbidden**
uw cannot block M	**forced**	**free**	irrelevant

Table 1: The three types of restricted edges are marked with bold letters. The columns tells edge uw's role regarding being in a matching, while the rows split cases based on uw's ability to block a matching.

uncertain preferences that mask an underlying strict order [30, 5, 4]. If two edges are in the same tie because of incomplete information derived from the agent, then super-stable matchings form the set of matchings that guarantee stability for all possible true preferences.

Another classical direction of research is to distinguish some of the edges based on their ability to be part of or to block a matching. Table 1 provides a structured overview of the three sorts of restricted edges that have been defined in earlier papers [20, 11, 12, 3, 22, 10]. The mechanism designer can specify three sets of restricted edges: *forced* edges must be in the output matching, *forbidden* edges must not appear in it, and finally, *free* edges cannot block the matching, regardless of the preference ordering.

The market designer's motivation behind forced and forbidden edges is clear. By adding these restricted edges to the instance, one can shrink the set of stable solutions to the matchings that contain a particularly important or avoid an unwelcome partnership between agents. Free edges model a less intuitive, yet ubiquitous scenario in applications [3]. Agents are often not aware of the preferences of others, not even once the matching has been specified. This typically occurs in very large markets, such as job markets [2], or if the preferences are calculated rather than just provided by the agents, such as in medical [8] and social markets [1]. Agents who cannot exchange their preferences are connected via a free edge. If a matching is only blocked by free edges, then no pair of agents can undermine the stability of it.

In this paper, we combine weakly ordered lists and restricted edges, and determine the computational complexity of finding a stable matching in all cases not solved yet.

1.1. Literature review

We first focus on the known results for the SMT problem without restricted edges, and then switch to the SM problem with edge restrictions. Finally, we list all progress up to our paper in SMT with restricted edges.

Ties. If all edges are unrestricted, a weakly stable matching always exists, because generating any linear extension to each preference list results in a classical SM instance, which admits a solution [14]. This solution remains stable in the original instance as well. On the other hand, strong and super-stable matchings are not guaranteed to exist. However, there are polynomial-time algorithms to output a strongly/super-stable matching or a proof for its nonexistence [16, 25].

Restricted edges. Dias et al. [11] showed that the problem of finding a stable matching in a SM instance with forced and forbidden edges or reporting that none exists is solvable in $O(m)$ time, where m is the number of edges in the instance. Approximation algorithms for instances not admitting any stable matching including all forced and avoiding all forbidden edges were studied in [10]. The existence of free edges can only enlarge the set of stable solutions, thus a stable matching with free edges always exists. However, in the presence of free edges, a maximum-cardinality stable matching is NP-hard to find [3]. Kwanashie [22, Sections 4 and 5] performed an exhaustive study on various stable matching problems with free edges. The term "stable with free edges" [8, 13] is equivalent to the adjective "socially stable" [3, 22] for a matching.

Ties and restricted edges. Table 2 illustrates the known and our new results on problems that arise when ties and restricted edges are combined in an instance. Weakly stable matchings in the presence of forbidden edges were studied by Scott [32], where the author shows that deciding whether a matching exists avoiding the set of forbidden edges is NP-complete. A similar hardness result was derived by Manlove et al. [26] for the case of forced edges, even if the instance has a single forced edge. Forced and forbidden edges in super-stable matchings were studied by Fleiner et al. [12], who gave a polynomial-time algorithm to decide whether a stable solution exists. Strong stability in the presence of forced and forbidden edges is covered by Kunysz [21], who gave a polynomial-time algorithm for the weighted strongly stable matching problem with non-negative edge weights. Since strongly stable matchings are always of the same cardinality [23, 18], a stable solution or a proof for its nonexistence can be found via setting the edge weights to 0 for forbidden edges, 2 for forced edges, and 1 for unrestricted edges.

Existence	weak	strong	super
forbidden	NP-complete [32] **even if $\vert P\vert = 1$**	$O(nm)$ [21]	$O(m)$ [12]
forced	NP-complete even if $\vert Q\vert = 1$ [26]	$O(nm)$ [21]	$O(m)$ [12]
free	always exists	**NP-complete**	**NP-complete**

Table 2: Previous and our results summarized in a table. The contribution of this paper is marked by bold gray font. The instance has n vertices, m edges, $|P|$ forbidden edges, and $|Q|$ forced edges.

1.2. Our contributions

In Section 3 we prove a stronger result than the hardness proof in [32] delivers: we show that finding a weakly stable matching in the presence of forbidden edges is NP-complete even if the instance has a single forbidden edge.

As a byproduct, we gain insight into the well-known maximum size weakly stable matching problem (without any edge restriction). This problem is known to be NP-complete [19, 26], even if preference lists are of length at most three [17, 27]. On the other hand, if the graph is complete, a complete weakly stable matching is guaranteed to exist. It turns out that this completeness is absolutely crucial to keep the problem tractable: as we show here, if the graph is a complete bipartite graph missing exactly one edge, then deciding whether a perfect weakly stable matching exists is NP-complete.

We turn to the problem of free edges under strong and super-stability in Section 4. We show that deciding whether a strongly/super-stable matching exists when free edges occur in the instance is NP-complete. This hardness is in sharp contrast to the polynomial-time algorithms for the weighted strongly/ super-stable matching problems. Afterwards, we show that deciding the existence of a strongly or super-stable matching in an instance with free edges is fixed-parameter tractable parameterized by the number of free edges.

2. Preliminaries

The input of the stable marriage problem with ties consists of a bipartite graph $G = (U \cup W, E)$ and for each $v \in U \cup W$, a weakly ordered preference list O_v of the edges incident to v. We denote the number of vertices in G by n, while m stands for the number of edges. An edge connecting vertices u and w is denoted by uw. We say that the preference lists in an instance are derived from a *master list* if there is a weak order O of $U \cup W$ so that each O_v where $v \in U \cup W$ can be obtained by deleting entries from O.

The set of restricted edges consists of the set of *forbidden edges* P, the set of *forced edges* Q, and the set of *free edges* F. These three sets are disjoint.

Definition 1. *A matching M is* weakly/strongly/super-stable with restricted edge sets P, Q, and F, *if $M \cap P = \emptyset$, $Q \subseteq M$, and the set of edges blocking M in a weakly/strongly/super sense is a subset of F.*

3. Weak stability

In Theorem 1 we present a hardness proof for the weakly stable matching problem with a single forbidden edge, even if this edge is ranked last by both end vertices. The hardness of the maximum-cardinality weakly stable matching problem in dense graphs (Theorem 2) follows easily from this result.

Problem 1. SMT-FORBIDDEN-1
Input: A complete bipartite graph $G = (U \cup W, E)$, a forbidden edge $P = \{uw\}$ and preference lists with ties.
Question: Does there exist a weakly stable matching M so that $uw \notin M$?

Theorem 1. SMT-FORBIDDEN-1 *is* NP*-complete, even if all ties are of length two, they appear only on one side of the bipartition and at the beginning of the complete preference lists, and the forbidden edge is ranked last by both its end vertices.*

Proof. SMT-FORBIDDEN-1 is clearly in NP, as any matching can be checked for weak stability in linear time.

We reduce from the PERFECT-SMTI problem defined below, which is known to be NP-complete even if all ties are of length two, and appear on one side of the bipartition and at the beginning of the preference lists, as shown by Manlove et al. [26].

Problem 2. PERFECT-SMTI
Input: An incomplete bipartite graph $G = (U \cup W, E)$, and preference lists with ties.
Question: Does there exist a perfect weakly stable matching M?

Construction. To each instance $\mathcal{I}$ of PERFECT-SMTI, we construct an instance $\mathcal{I}'$ of SMT-FORBIDDEN-1.

Let $G = (U \cup W, E)$ be the underlying graph in instance $\mathcal{I}$. When constructing G' for $\mathcal{I}'$, we add two men u_1 and u_2 to U, and two women w_1 and w_2 to W. On vertex classes $U' = U \cup \{u_1, u_2\}$ and $W' = W \cup \{w_1, w_2\}$, G' will be a complete bipartite graph. As the list below shows, we start with the original edge set $E(G)$ in stage 0, and then add the remaining edges in four further stages. An example for the built graph is shown in Figure 1.

0. $E(G)$
   We keep the edges in $E(G)$ and also preserve the vertices' rankings on them. These edges are solid black in Figure 1.

1. $(U \times \{w_1\}) \cup (\{u_1\} \times W)$
   We first connect u_1 to all women in W, and w_1 to all men in U. Man u_1 (woman w_1) ranks the women from W (men from U) in an arbitrary order. Each $u \in U$ ($w \in W$) ranks w_1 (u_1) after all their edges in $E(G)$. These edges are loosely dashed green in Figure 1.

2. $(U \times W) \setminus E(G)$
   Now we add for each pair $(u, w) \in U \times W$ with $uw \notin E(G)$ the edge uw, where u (w) ranks w (u) even after w_1 (u_1). These edges are densely dashed blue in Figure 1.

3. $\big[(U \cup \{u_1\}) \times \{w_2\}\big] \cup \big[\{u_2\} \times (W \cup \{w_1\})\big]$
   Man u_2 is connected to all women from $W \cup \{w_1\}$, and ranks all these women in an arbitrary order. The women from $W \cup \{w_1\}$ rank u_2 worse than any already added edge. Similarly, w_2 is connected to all men from $M \cup \{u_1\}$, and ranks all these men in an arbitrary order. The men from $M \cup \{u_1\}$ rank w_2 worse than any already added edge. These edges are dotted red in Figure 1.

4. u_1w_1 and u_2w_2
   Finally, we add the edges u_1w_1 and u_2w_2, which are ranked last by both of their end vertices. Edge u_2w_2 is the only forbidden edge and it is the violet zigzag edge in Figure 1, while u_2w_2 is wavy gray.

Claim: $\mathcal{I}$ admits a perfect weakly stable matching if and only if $\mathcal{I}'$ admits a weakly stable matching not containing u_2w_2.

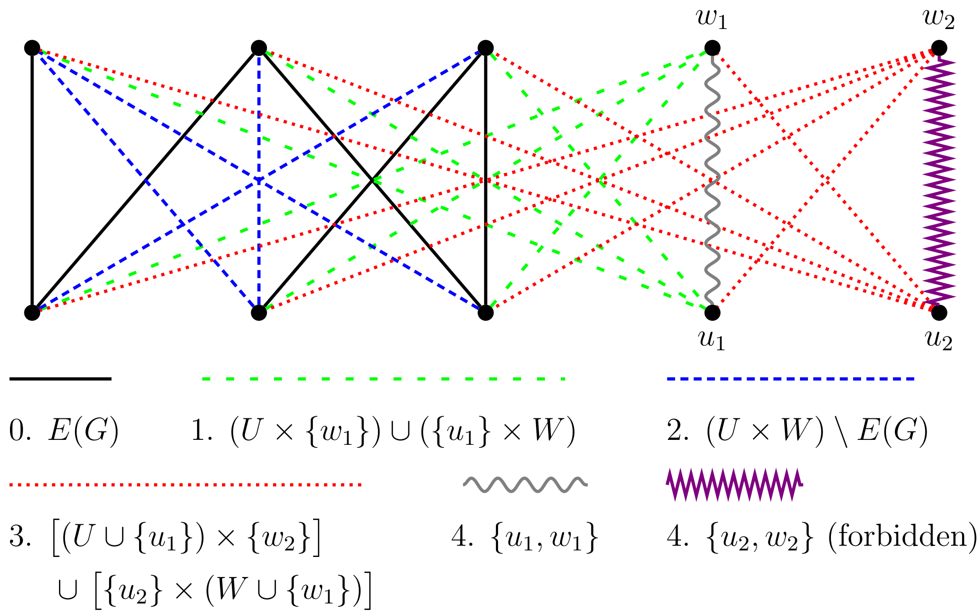

Figure 1: An example for the reduction. The legend below the graph lists the six groups of edges in the preference order at all vertices. The edges from the PERFECT-SMTI instance (drawn in solid black) keep their ranks. Every vertex ranks solid black edges best, then loosely dashed green edges, then densely dashed blue edges, then dotted red edges, then the wavy gray edge $\{u_1, w_1\}$ and the forbidden violet zigzag edge $\{u_w, w_2\}$.

($\Rightarrow$) Let M be a perfect weakly stable matching in $\mathcal{I}$. We construct M' as $M \cup \{u_1 w_2\} \cup \{u_2 w_1\}$. Clearly, M' is a matching not containing the forbidden edge $u_2 w_2$, so it only remains to show that M' is weakly stable. We do this by case distinction on a possible weakly blocking edge.

0. $E(G)$
   Since M does not admit a weakly blocking edge in $\mathcal{I}$, no edge from the original $E(G)$ can block M' weakly in $\mathcal{I}'$.

1. $(U \times \{w_1\}) \cup (\{u_1\} \times W)$
   All vertices in $U \cup W$ rank these edges lower than their edges in M'.

2. $(U \times W) \setminus E(G)$
   Edges in this set cannot block M' weakly because they are ranked worse than edges in M' by both of their end vertices.

3. $\big[(U \cup \{u_1\}) \times \{w_2\}\big] \cup \big[(W \cup \{w_1\}) \times \{u_2\}\big]$
   Vertices in $U \cup W$ prefer their edge in M' to all edges in this set. Since they are in M', u_1w_2 and u_2w_1 also cannot block M' weakly.

4. u_1w_1 and u_2w_2
   These two edges are strictly worse than $u_1w_2 \in M'$ and $u_2w_1 \in M'$ at all four end vertices.

($\Leftarrow$) Let M' be a weakly stable matching in $\mathcal{I}'$ and $u_2w_2 \notin M'$. Since G' is a complete bipartite graph with the same number of vertices on both sides, M' is a perfect matching. In particular, u_2 and w_2 are matched by M', say to w and u, respectively. Since M' does not contain the forbidden edge u_2w_2, we have that $u \neq u_2$ and $w \neq w_2$. Then we have $w = w_1$ and $u = u_1$, as uw blocks M' weakly otherwise.

If M' contains an edge $uw \notin E(G)$ with $u \in U$ and $w \in W$, then this implies that uw_1 is a weakly blocking edge. Thus, $M := M' \setminus \{u_1w_2, u_2w_1\} \subseteq E(G)$, i.e. it is a perfect matching in G. This M is also weakly stable, as any weakly blocking edge in G immediately implies a weakly blocking edge for M', which contradicts our assumption on M' being a weakly stable matching. □

As a byproduct, we get that MAX-SMTI-DENSE, the problem of deciding whether an almost complete bipartite graph admits a perfect weakly stable matching, is also NP-complete.

Problem 3. MAX-SMTI-DENSE
Input: A bipartite graph $G = (U \cup W, E)$, where $E(G) = \{uw : u \in U, w \in W\} \setminus \{u^*w^*\}$ for some $u^* \in U$ and $w^* \in W$, and preference lists with ties.
Question: Does there exist a perfect weakly stable matching M?

Theorem 2. MAX-SMTI-DENSE *is* NP*-complete, even if all ties are of length two, are on one side of the bipartition, and appear at the beginning of the preference lists.*

Proof. MAX-SMTI-DENSE is in NP, as a matching can be checked for stability in linear time.

We reduce from SMT-FORBIDDEN-1. By Theorem 1, this problem is NP-complete even if the forbidden edge uw is at the end of the preference lists of u and w. For each such instance $\mathcal{I}$ of SMT-FORBIDDEN-1, we construct an instance $\mathcal{I}'$ of MAX-SMTI-DENSE by deleting the forbidden edge uw.

Claim: The instance $\mathcal{I}$ admits a weakly stable matching if and only if $\mathcal{I}'$ admits a perfect weakly stable matching.

($\Rightarrow$) Let M be a weakly stable matching for $\mathcal{I}$. As SMT-FORBIDDEN-1 gets a complete bipartite graph as an input, M is a perfect matching. Since M does not contain the edge uw, it is also a matching in $\mathcal{I}'$. Moreover, M is weakly stable there, because the transformation only removed a possible blocking edge and added none of these.

($\Leftarrow$) Let M' be a perfect weakly stable matching in $\mathcal{I}'$. Since uw is at the end of the preference lists of u and w, and M' is perfect, uw cannot block M'. Thus, M' is weakly stable in $\mathcal{I}$. □

Having shown a hardness result for the existence of a weakly stable matching even in very restricted instances with a single forbidden edge in Theorem 1, we now turn our attention to strongly and super-stable matchings.

4. Strong and super-stability

As already mentioned in Section 1.1, strongly and super-stable matchings can be found in polynomial time even if both forced and forbidden edges occur in the instance [12, 21]. Thus we consider the case of free edges, and in Theorem 3 and Proposition 4 we show hardness for the strong and super-stable matching problems in instances with free edges. The same construction suits both cases. Then, in Proposition 5 we remark that both problems are fixed-parameter tractable with the number of free edges $|F|$ as the parameter.

Problem 4. SSMTI-FREE
Input: A bipartite graph $G = (U \cup W, E)$, a set $F \subseteq E$ of free edges, and preference lists with ties.
Question: Does there exist a matching M so that $uw \in F$ for all $uw \in E$ that blocks M in the strongly/super-stable sense?

In SSMTI-FREE, we define two problem variants simultaneously, because all our upcoming proofs are identical for both of these problems. For the super-stable marriage problem with ties and free edges, all super-blocking edges must be in F, while for the strongly stable marriage problem with ties and free edges, it is sufficient if a subset of these, the strongly blocking edges, are in F.

Theorem 3. SSMTI-FREE *is* NP*-complete even in graphs with maximum degree four, and if preference lists of women are derived from a master list.*

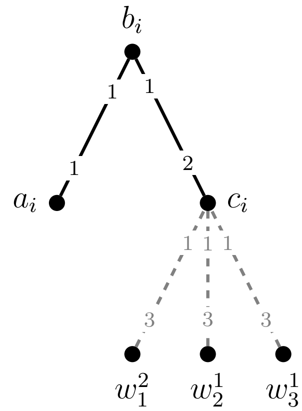

Figure 2: An example of a clause gadget for the clause C_i, containing the variables x_1, x_4, and x_5. The interconnecting edges are dashed and gray.

Proof. SSMTI-FREE is clearly in NP because the set of edges blocking a matching can be determined in linear time.

We reduce from the 1-IN-3 POSITIVE 3-SAT problem, defined below, which is known to be NP-complete [31, 15, 28].

Problem 5. 1-IN-3 POSITIVE 3-SAT
Input: A 3-SAT formula, in which no literal is negated and every variable occurs in at most three clauses.
Question: Does there exist a satisfying truth assignment that sets exactly one literal in each clause to be true?

Construction. To each instance $\mathcal{I}$ of 1-IN-3 POSITIVE 3-SAT, we construct an instance $\mathcal{I}'$ of SSMTI-FREE.

Let $x_1, \ldots, x_n$ be the variables and $C_1, \ldots, C_m$ be the clauses of the 1-IN-3 POSITIVE 3-SAT instance $\mathcal{I}$. For each clause C_i, we add a clause gadget consisting of three vertices a_i, b_i, and c_i, where b_i is connected to a_i and c_i, as shown in Figure 2. While vertices a_i and b_i do not have any further edge, c_i will be incident to three *interconnecting* edges leading to variable gadgets. Vertex b_i is ranked first by a_i and last by c_i, and these two vertices are placed in a tie by b_i.

For each variable x_i, occurring in the three clauses C_{i_1}, C_{i_2}, and C_{i_3}, we add a variable gadget with nine vertices y_i^j, z_i^j, and w_i^j for $j \in [3]$, as indicated in Figure 3. Each vertex z_i^j is connected only to y_i^j by a free edge, and these

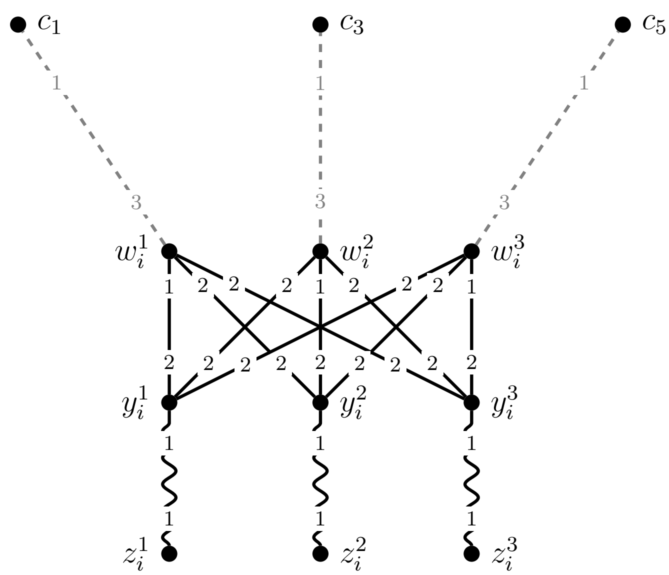

Figure 3: An example of a variable gadget for the variable x_i occurring, where x_i occurs exactly in the clauses C_1, C_3, and C_5. Free edges are marked by wavy lines, while interconnecting edges are dashed and gray.

are the only free edges in our construction. For each $(\ell, j) \in [3]^2$, we add an edge $w_i^\ell y_i^j$, which is ranked second (after z_i^j) by y_i^j. The vertex w_i^ℓ ranks this edge at position one if $\ell = j$ and else at position two. Finally, we connect the vertex w_i^ℓ to the vertex c_{i_ℓ} by an interconnecting edge, ranked at position one by c_{i_ℓ} and position three by w_i^ℓ.

The resulting instance is bipartite: $U = \{z_i^j, w_i^j, b_i\}$ is the set of men and $W = \{y_i^j, c_i, a_i\}$ is the set of women. One easily sees that the maximum degree in our reduction is four.

Note that the preference lists of the women in the SSMTI-FREE instance are derived from a master list. The master list for the women $W = \{y_i^j, c_i, a_i\}$ is the following. At the top are all vertices of the form $\{z_i^j\}$ in a single tie, followed by all vertices of the form $\{w_i^j\}$ in a single tie, and finally, all other vertices $\{b_i\}$ at the bottom of the preference list, in a single tie.

Claim: $\mathcal{I}$ is a YES-instance if and only if $\mathcal{I}'$ admits a strongly/super-stable matching.

($\Rightarrow$) Let T be a satisfying truth assignment such that for each clause, exactly one literal is true. For each true variable x_i in this assignment, let M

contain the edges $w_i^\ell c_{i_\ell}$ and $y_i^\ell z_i^\ell$ for each $\ell \in [3]$. For all other variables, let M contain $w_i^\ell y_i^\ell$ for each $\ell \in [3]$. For each clause C_i, add the edge $a_i b_i$ to M.

Following these rules, we have constructed a matching. It remains to check that M is super-stable (and thus also strongly stable). Since a_i is matched to its only neighbor, it cannot be part of a super-blocking edge. Since each c_i is matched along an interconnecting edge, which is better than b_i, no super-blocking edge involves b_i. A super-blocking interconnecting edge $c_i w_j^\ell$ implies that w_j^ℓ is not matched to any y_j^ℓ, however this is only true if $c_i w_j^\ell \in M$. A super-blocking edge $w_i^\ell y_i^j$ does not appear. Either w_i^ℓ is matched to its unique first choice y_i^ℓ and therefore not part of a super-blocking edge, or y_i^j is matched to its unique first choice z_i^j, and thus, y_i^j is not part of a super-blocking edge.

($\Leftarrow$) Let M be a strongly stable matching (note that any super-stable matching is also strongly-stable). Then M contains the edge $a_i b_i$, and c_i is matched to a vertex w_j^ℓ for all $i \in [m]$, as else $c_i b_i$ or $a_i b_i$ blocks M strongly. If $w_j^\ell c_i \in M$, then $y_j^a z_j^a \in M$ for all $a \in [3]$, as else $w_j^\ell y_j^a$ would be a strongly blocking edge. This, however, implies that $w_j^a c_{j_a} \in M$ for all $a \in [3]$, as else $w_j^a c_{j_a}$ would be a strongly blocking edge.

Thus, for each variable x_i, the matching M contains either all edges $w_i^\ell c_{i_\ell}$ for $\ell \in [3]$ or none of these edges. Thus, the variables x_i such that M contains $w_i^\ell c_{i_\ell}$ for $\ell \in [3]$ induce a truth assignment such that for each clause, exactly one literal is true. □

This proof aimed at the hardness of the restricted case, in which the underlying graph has a low maximum degree. For the sake of completeness, we add another variant, which is defined in a complete bipartite graph.

Proposition 4. SSMTI-FREE *is* NP*-complete, even in complete bipartite graphs, where each tie has length at most three.*

Proof. We reduce from SSMTI-FREE. Given a SSMTI-FREE instance in graph G, we add all non-present edges between men and women as free edges, ranked worse than any edge from $E(G)$. We call the resulting graph H.

Clearly, a strongly/super-stable matching in G is also strongly/super-stable in H, as we only added free edges.

Vice versa, let M be a strongly/super-stable matching in H. Let $M' := M \cap E(G)$ arise from M by deleting all edges not in $E(G)$. Then M' clearly is a matching in G, so it remains to show that M' is strongly/super-stable.

Assume that there is a blocking edge uw in G, in the strongly/super-stable sense. Since uw is not blocking in H, at least one of u and w has to be matched in H, but not in G. However, this vertex prefers uw also to its partner in H, and thus, uw is also blocking in H, which is a contradiction. □

Note that SSMTI-FREE becomes polynomial-time solvable if only a constant number of edges is free in the same way as MAX-SSMI, the problem of finding a maximum-cardinality stable matching with strict lists and free edges [3].

Proposition 5. SSMTI-FREE *can be solved in* $\mathcal{O}(2^k nm)$ *time in the strongly stable case, and in* $\mathcal{O}(2^k m)$ *time in the super-stable case, where* $k := |F|$ *is the number of free edges,* $n := |V(G)|$ *is the number of vertices, and* $m := |E(G)|$ *is the number of edges.*

Proof. For each subset $Q \subseteq F$ of free edges, we construct an instance of SSMTI-FORCED as follows. Mark all edges in Q as forced, and delete all edges in $F \setminus Q$.

If any of the SSMTI-FORCED instances admits a stable matching, then this is clearly a stable matching in the SSMTI-FREE instance, as only free edges were deleted. Vice versa, any solution M for the SSMTI-FREE instance containing exactly the set of forced edges Q (i.e. $Q = M \cap F$) immediately implies a solution for the SSMTI-FORCED instance with forced edges Q.

Clearly, there are 2^k subsets of F. Since any instance of SSMTI-FORCED can be solved in $\mathcal{O}(nm)$ time in the strongly stable case [12] and in $\mathcal{O}(m)$ time in the super-stable case [21], the running time follows. □

5. Conclusion

Studying the stable marriage problem with ties combined with restricted edges, we have shown three NP-completeness results. Our computational hardness results naturally lead to the question whether imposing master lists on both sides makes the problems easier to solve. Moreover, it is open whether SMT-FORBIDDEN-1 remains hard in bounded-degree graphs. In addition, one may try to identify relevant parameters for our problems and then decide whether they are fixed-parameter tractable or admit a polynomial-sized kernel with respect to these parameters.

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