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**Spectral risk measure
of holding stocks in the long run**

ZSOLT BIHARY – PÉTER CSÓKA – DÁVID ZOLTÁN SZABÓ

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Spectral risk measure of holding stocks in the long run

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Spectral risk measure of holding stocks in the long run

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Abstract

We investigate how the spectral risk measure associated with holding stocks rather than a risk-free deposit, depends on the holding period. Previous papers have shown that within a limited class of spectral risk measures, and when the stock price follows specific processes, spectral risk becomes negative at long periods. We generalize this result for arbitrary exponential Lévy processes. We also prove the same behavior for all spectral risk measures (including the important special case of Expected Shortfall) when the stock price grows realistically fast and when it follows a Geometric Brownian Motion or a Finite Moment Log Stable process. This result would suggest that holding stocks for long periods has a vanishing risk. However, using realistic models, we find numerically that the risk increases for a few decades and reaches zero at around 100 years. Therefore, we conclude that holding stocks is risky for all practically relevant periods.

Keywords: Coherent Risk Measures, Spectral Risk Measures, Lévy processes, Finite Moment Log Stable Model, Time Diversification

JEL: G11

A részvénytartás spektrális kockázata hosszú távon

Bihary Zsolt – Csóka Péter – Szabó Dávid Zoltán

Összefoglaló

Tanulmányunkban azt vizsgáljuk: hogyan függ a tartási periódustól annak a spektrális kockázata, hogy kockázatmentes bankbetét helyett részvényekbe fektetünk. A korábbi kutatásokban megmutatták, hogy bizonyos spektrális kockázati mértékekre és folyamatokra a spektrális kockázat negatív lesz hosszú távon. Mi ezt az eredményt általánosítjuk tetszőleges exponenciális Lévy-folyamatra. Belátjuk ugyanezt tetszőleges spektrális kockázati mértékre (beleértve az expected shortfall esetét) is, ha a részvényárfolyam realiztikusan gyorsan nő és az geometriai Brown-mozgást végez, vagy véges momentumú log-stabil folyamatot követ. Ez az eredmény azt sugallja, hogy a részvénytartás kockázata hosszú távon elenyésző. Ugyanakkor realiztikus modelleket használva numerikusan azt találjuk, hogy a kockázat több évtizeden át növekszik és csak kb. 100 év után éri el a nullát. Ezért konklúziónk az, hogy minden praktikusán releváns időtávra a részvénytartás kockázatos.

Tárgyszavak: Koherens kockázati mértékek, spektrális kockázati mértékek, Lévy-folyamatok, időbeli diverzifikáció

JEL: G11

Spectral risk measure of holding stocks in the long run*

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June 19, 2018

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We investigate how the spectral risk measure associated with holding stocks rather than a risk-free deposit, depends on the holding period. Previous papers have shown that within a limited class of spectral risk measures, and when the stock price follows specific processes, spectral risk becomes negative at long periods. We generalize this result for arbitrary exponential Lévy processes. We also prove the same behavior for all spectral risk measures (including the important special case of Expected Shortfall) when the stock price grows realistically fast and when it follows a Geometric Brownian Motion or a Finite Moment Log Stable process. This result would suggest that holding stocks for long periods has a vanishing risk. However, using realistic models, we find numerically that the risk increases for a few decades and reaches zero at around 100 years. Therefore, we conclude that holding stocks is risky for all practically relevant periods.

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1 Introduction

There has been tremendous interest both from practitioners and researchers on the question of how risky it is to hold stocks rather than a risk-free deposit in the long run. The existence of time diversification, meaning that the risk of holding stocks decreases with the time horizon, has profound effects on all long-term investments such as pension savings and target date funds. Industry practitioners hold the common wisdom that the longer the intended holding period for a portfolio, the more it should contain risky, rather than risk-free investments.

The academic literature on the topic is more controversial (Bennyhoff, 2009). Siegel (1998) showed that for investment horizons of at least 15 years, the risk of holding stocks (measured by realized variance) is lower than the risk of holding bonds or Treasury bills, and the risk of holding stocks is decreasing over time. However, Pástor and Stambaugh (2012) argue that stocks are more volatile in the long run from the perspective of an investor. The reason is that investors do not know the parameters of the stock price process (especially not the conditional expected return), and by making noisy predictions, at a 50-year horizon they face about 1.3 times higher return variance per year than at a 1-year horizon. Modeling the investors' beliefs, Avramov, Cederburg, and Lučivjanská (2017) find that there could be investors who perceive that the mean reversion of stocks is weaker, and hence they consider stocks riskier.

One can consider the probability that at time t the stock investment value $S(t)$ is smaller than the risk free deposit value S_0e^{rt} , where S_0 is the initial investment and r is a constant risk-free rate. This probability decreases with t , supporting the notion that the risk of the stock investment decreases with the holding period. However, as noted by Harlow (1991), this concept does not take into account the difference between S_0e^{rt} and $S(t)$, and this difference can, of course, be larger and larger as t increases.

Bodie (1995) approached the concept of risk differently by quantifying risk as the price of a put option that guarantees the risk-free payout S_0e^{rt} at time t . The price of the corresponding put option was found to be increasing as t increases, which can be interpreted as the risk of holding stocks increases over time. Wilkie (2001) criticized the methodology of Bodie (1995) by saying that if we wish to guarantee that we get at least the proceeds of a risk-free investment, then (since buying the option also has some costs) we have to invest everything in the risk-free investment, and pricing the option makes no sense. Moreover, Ferguson and Dean (1996) argued that the method of Bodie (1995) forces one to conclude that the risk of an asset relative to the risk of any other asset increases with the investment horizon, leading to a paradox.

Following Treussard (2006) and Nguyen, Pham, and Tran (2012), we consider the risk of holding a stock rather than a risk-free deposit over time. More precisely, we analyze the spectral risk measure (Acerbi, 2002) of the loss variable $Y(t) = S_0e^{rt} - S(t)$. If $Y(t)$ is positive (negative), then the stock performs poorly (well), in comparison with the risk free money market account. Spectral risk measures are coherent (Artzner, Delbaen, Eber, and Heath, 1999) and can be expressed as a weighted average of losses, with increasing weights for higher losses. Thus we can think of the spectral risk of $Y(t)$ as a weighted average regret value associated with having bought the stock, rather than investing in a money market account.

As a special spectral risk, we consider Expected Shortfall (ES) (Acerbi and Tasche,

2002; Rockafellar and Uryasev, 2002), which can be computed and estimated efficiently (Acerbi and Tasche, 2002) and is backtestable (Acerbi and Szekely, 2014). Moreover, in 2013 the Basel Committee of Banking Supervision took the decision (Basel Committee, 2013) of using ES for capital adequacy internal models, underscoring the importance of ES as a practical risk measure.

To describe the possible evolution of stock prices, we consider different stochastic processes. Besides the standard geometric Brownian motion (GBM) model, we analyze the exponential Lévy model (see for instance Mandelbrot (1963), Bertoin (1996), and McCulloch (1996)). The exponential Lévy process, of which GBM is a special case, offers more flexibility to model the heavy tail behavior of stock prices. As an illustration, we will investigate a particular exponential Lévy subclass called the Finite Moment Log Stable (FMLS) model (Carr and Wu, 2003). The FMLS model can capture heavier tails for the losses.

There are two distinct regimes where the long-time behavior of risk is different. Depending on the parameters of the stock price process (the drift μ and the convexity adjustment κ), we distinguish between a high growth regime ($\mu > r + \kappa$) and a medium growth regime ($r + \kappa \geq \mu > r$). Our results extend those of Nguyen, Pham, and Tran (2012), since we treat the exponential Lévy model in general and the FMLS model in particular, and we have results for Expected Shortfall both in the GBM and in the FMLS models. We prove in case of a GBM stock process that in the high growth regime all spectral risk measures (including Expected Shortfall) will be negative at high t . We also investigate general exponential Lévy models and provide conditions under which in both at the high growth and at the medium growth regimes spectral risk will be negative at high t . For the FMLS model, we obtain the same results as for the GBM model, that is, in the high growth regime all spectral risk measures will be negative at high t .

This result would suggest that holding stocks for long periods has a vanishing risk. However, using realistic price process models, we find numerically that the risk is increasing for about 30 years and starts to decrease only after that, reaching zero at around 100 years. After performing some sensitivity tests, we conclude that holding stocks is risky for all practically relevant periods.

The structure of the paper is as follows. Spectral risk measures and their Choquet representation are given in Section 2. Section 3 contains the investigated stochastic processes, Section 4 provides the analytical results. Section 5 contains numerical results, Section 6 concludes.

2 Spectral risk measures

Coherent risk measures (Artzner, Delbaen, Eber, and Heath, 1999) have a sound axiomatic foundation, are supported by general equilibrium theory (Csóka, Herings, and Kóczy, 2007) and they can also capture liquidity risk (Acerbi and Scandolo, 2008). Consider the random variable X that represents the loss distribution of a portfolio at time t . Thus positive values of X are representing losses, negative values of X are representing profits. The set of the random loss variables is denoted by \mathcal{X} .

Definition 2.1. The risk measure $\rho : \mathcal{X} \rightarrow \mathbb{R}$ is called a *coherent risk measure* if it satisfies the following properties:

- monotonicity: $Y \leq X \Rightarrow \rho(Y) \leq \rho(X)$,
- positive homogeneity: $h > 0 \Rightarrow \rho(hX) = h\rho(X)$,
- sub-additivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$,
- translation invariance: $a \in \mathbb{R} \Rightarrow \rho(X + a) = \rho(X) + e^{-rt}a$,

where e^{-rt} is the discount factor at time t .

Throughout this paper we will assume a constant risk-free rate r . The discount factor e^{-rt} is omitted in some papers, but it is especially important for us here, as we will investigate long horizons.

As a prominent coherent risk measure, Expected Shortfall (ES) (Acerbi and Tasche, 2002; Rockafellar and Uryasev, 2002) can be computed and estimated efficiently (Acerbi and Tasche, 2002), and it is backtestable (Acerbi and Szekely, 2014). Moreover, in 2013 the Basel Committee of Banking Supervision took the decision (Basel Committee, 2013) of using ES for capital adequacy internal models. Given a random loss variable $X \in \mathcal{X}$ and a confidence level $\alpha \in [0, 1]$, $\text{ES}_\alpha(X)$ is the average of the loss in the worst $100(1-\alpha)\%$ cases. As an example, $\alpha = 90\%$, and $\text{ES}_{90\%}$ is the average of the loss in the worst 10% of the cases.

To define ES formally in our setting, we need some more notation. Given a random loss variable $X \in \mathcal{X}$, let $F_X(x) = P[X \leq x]$ denote its distribution function. The generalized inverse of the loss distribution X is defined as $F_X^{-1}(p) = \inf\{x | F_X(x) \geq p\}$.

Definition 2.2. Given a random loss variable $X \in \mathcal{X}$, for $\alpha \in [0, 1]$, the *Expected Shortfall of X at α* , $\text{ES}_\alpha(X)$ is defined as

$$\text{ES}_\alpha(X) = e^{-rt} \frac{1}{1-\alpha} \int_\alpha^1 F_X^{-1}(p) dp.$$

For $\alpha = 1$, ES_1 , that is the (discounted) maximum loss is defined as

$$\text{ES}_\alpha(X) = e^{-rt} \text{ess. sup}\{X\}.$$

Spectral risk measures (Acerbi, 2002) are also coherent and they are generalizing ES to get the weighted average of the losses with increasing weights as follows.

Definition 2.3. Given a random loss variable $X \in \mathcal{X}$, the *spectral risk measure of X* , $\rho_\phi(X)$ is defined as

$$\rho_\phi(X) = e^{-rt} \int_0^1 F_X^{-1}(p) \phi(p) dp,$$

where $\phi \in L^1([0, 1])$, and the following properties are true:

- ϕ is positive;
- ϕ is monotonically increasing;
- $\int_0^1 |\phi(p)| dp = 1$.

Note again that all the usual formulas are written for losses, not gains, and they are adjusted with the e^{-rt} discount factor. As it is clear from its definition, ES is a spectral risk measure that we will use as a specific example in our numerical calculations.

We will work with spectral risk measures throughout this paper, but we will use an equivalent definition derived as a special case of Choquet integrals (Choquet, 1954). Choquet integral risk measures (see for instance Sriboonchita, Wong, Dhompongsa, and Nguyen (2009)) are defined as follows.

Definition 2.4. Given a random loss variable $X \in \mathcal{X}$, the *Choquet integral risk measure* of X , $\rho_h(X)$ is given by

$$\rho_h(X) = e^{-rt} \int_0^\infty h(1 - F_X(x)) dx + e^{-rt} \int_{-\infty}^0 [h(1 - F_X(x)) - 1] dx,$$

where h , the so-called distortion function (Wang, 1996) is non-decreasing and satisfies $h(0) = 0$ and $h(1) = 1$.

Nguyen, Pham, and Tran (2012) notes that for concave distortion functions h there exists a connection between Choquet integral risk measures and spectral risk measures such that if $h'(1 - p) = \phi(p)$, then $\rho_h(X) = \rho_\phi(X)$ for all $X \in \mathcal{X}$. In this work, we will refer to spectral risk measures and Choquet integral risk measures interchangeably. A distortion function h is non-degenerate, if there exists $p > 0$ such that $h'(p) > 0$, or, equivalently, there exists $q < 1$ such that $\phi(q) > 0$.

Following Treussard (2006) and Nguyen, Pham, and Tran (2012), we consider $Y(t) = S_0 e^{rt} - S(t)$ as the loss variable whose spectral risk we want to calculate using Definition 2.4. If $Y(t)$ is positive (negative), then the stock performs poorly (well), in comparison with the risk-free deposit. We can also think of $Y(t)$ as a regret value associated with having bought the stock, rather than holding our investment in a risk-free deposit.

Throughout the paper, we will restrict our attention to non-degenerate distortion functions. In practice, this means that we exclude the degenerate maximum loss risk measure.

3 Investigated models

Let $S(t)$ be a positive valued stochastic process which represents the stock price on the market. As usual, this stochastic process is assumed to be defined with respect to a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In the *geometric Brownian motion model (GBM)*, the stock price follows the stochastic process

$$S(t) = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}, \quad S(0) = S_0, \quad (1)$$

where $\mu > 0$ is the drift, $\sigma > 0$ is the volatility, and W_t is a Wiener process.

The standardized log return of this process $S(t)$ is distributed as

$$\frac{\ln\left(\frac{S(t)}{S_0}\right) - (\mu - \sigma^2/2)t}{\sigma\sqrt{t}} \sim \Phi[z], \quad (2)$$

where $\Phi[z]$ is the standard normal distribution function.

Exponential Lévy processes (see for instance Mandelbrot (1963), Bertoin (1996), and McCulloch (1996)), of which GBM is a special example, offer more flexibility to model heavier tail behaviour. In the exponential Lévy model, the stock price follows the stochastic process

$$S(t) = S_0 e^{(\mu - \kappa_X)t + X_t}, \quad S(0) = S_0, \quad (3)$$

where X_t is a Lévy process fully determined by its Lévy triplet \mathbb{T} , $\mu > 0$ is the drift, and κ_X is the convexity adjustment term that can be calculated as

$$\kappa_X = \frac{1}{t} \ln \mathbb{E} e^{X_t}. \quad (4)$$

The convexity adjustment κ_X plays the role of a compensator in the sense that $e^{-\kappa_X t + X_t}$ is a martingale. Note that κ_X does not depend on time. We will denote the distribution function of the Lévy process at a given time t by F_t .

We will consider a specific exponential Lévy model, called *Finite Moment Log Stable (FMLS)* model. In the FMLS model, the stock price movement follows the stochastic process (Carr and Wu, 2003)

$$S(t) = S_0 e^{(\mu - \kappa)t + \sigma L_t^\alpha}, \quad S(0) = S_0, \quad (5)$$

where μ is the drift, σ is the scale of volatility, L_t^α is alpha stable distributed with shape parameters $\alpha \in (1, 2)$ and $\beta = -1$, with scale parameter $t^{\frac{1}{\alpha}}$, and with zero mean, that is $L_t^\alpha \sim S(\alpha, -1, t^{\frac{1}{\alpha}}, 0)$ (see more details in Carr and Wu (2003)). Note that the only remaining free parameter for the distribution is α . The $\alpha \rightarrow 2$ limit corresponds to the GBM model. Decreasing α results in heavier tails for the losses. As before, κ is the convexity adjustment term for the FMLS model, given as

$$\kappa = -\sigma^\alpha \sec(\pi\alpha/2). \quad (6)$$

Note that κ is positive due to the properties of the *sec* function, as $\alpha \in (1, 2)$. Now the log return of the $S(t)$ process becomes

$$\ln \left(\frac{S(t)}{S_0} \right) = (\mu - \kappa)t + \sigma L_t^\alpha, \quad (7)$$

therefore its distribution is $S(\alpha, -1, \sigma t^{\frac{1}{\alpha}}, (\mu - \kappa)t)$ for a fixed t . Stable distributions have the following important property (see for example page 24 of Janicki and Weron (1994)).

Proposition 3.1. *If X is an alpha stable distributed random variable $X \sim S(\alpha, \beta, 1, 0)$ then $\gamma X + \delta \sim S(\alpha, \beta, \gamma, \delta)$ for $\delta > 0$ and $\gamma \in \mathbb{R}$.*

Using Proposition 3.1 we obtain

$$\frac{\ln \left(\frac{S(t)}{S_0} \right) - (\mu - \kappa)t}{\sigma t^{\frac{1}{\alpha}}} \sim S(\alpha, -1, 1, 0). \quad (8)$$

We denote the distribution and density functions of $S(\alpha, -1, 1, 0)$ at z by $\Theta_\alpha(z)$ and $\Theta'_\alpha(z)$, respectively.

4 Analytical Results

The main question of this paper is the long horizon behavior of $\rho_h(Y(t))$. As will be demonstrated in Section 5, it starts at zero, and at first, it typically grows. The question is whether it starts to decrease at a certain time or keeps rising. If $\rho_h(Y(t))$ becomes negative for a large t , it must have decreased from its positive maximum. In this case, it keeps decreasing below zero.

We will work under the statistical measure in this paper, but it will always be possible to consider results in the risk neutral measure with the substitution $\mu = r$. There are two distinct regimes where the long-time behavior of risk is different. The case when $\mu > r + \kappa$ we call the *high growth regime*. When $r + \kappa \geq \mu > r$, we are in the *medium growth regime*. The case $\mu \leq r$ (low growth regime) is not realistic, as it would mean that the risky asset grows slower on average than the risk-free asset.

Nguyen, Pham, and Tran (2012) investigated two exponential Lévy Processes, GBM and exponential Poisson. They proved that $h' > c > 0$ is a sufficient condition in both regimes for the risk to be negative at high t . In the high growth regime, they also proved this behavior if h' is continuous. We reinvestigated the GBM model and proved the new result that in the high growth regime all spectral risk measures would be negative at high t . Moreover, for general exponential Lévy models, we proved that in both high growth and medium growth regimes if $h' > c > 0$, then spectral risk will be negative at high t . We also investigated a specific exponential Lévy model, the FMLS model, and we obtained the same results as for the GBM model, that is in the high growth regime all spectral risk measures will be negative at high t .

4.1 GBM Model

In this section, we assume that the stock process $S(t)$ is a geometric Brownian motion.

The formula derived by Nguyen, Pham, and Tran (2012) for the spectral risk, adjusted to our definition that takes into account discounting, is:

$$\rho_h(Y(t)) = S_0[1 - e^{(\mu-r)t} \int_{-\infty}^{\infty} h'(\Phi[z]) \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}(z-\sigma\sqrt{t})^2} dz]. \quad (9)$$

We have the following own result.

Theorem 4.1. *In the GBM stock price model, $\rho_h(Y(t))$ is negative for a large t if $\mu > r + \frac{\sigma^2}{2}$.*

Proof. Since h' is a monotonically decreasing function in a closed interval, it can only have a countable number of jump points, so the function is almost surely continuous everywhere.

Let $A = [0, d]$. Due to the assumption of the theorem

$$h'(x) \geq c \quad \forall \quad x \in A, \quad (10)$$

where $c > 0$. For the integral in (9), we obtain the following lower bound.

$$\begin{aligned}
& \int_{-\infty}^{\infty} h'(\Phi[z]) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2} dz \\
&= \int_{\Phi^{-1}A} h'(\Phi[z]) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2} dz + \int_{R \setminus \Phi^{-1}A} h'(\Phi[z]) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2} dz \\
&\geq c \int_{\Phi^{-1}A} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2} dz + \int_{R \setminus \Phi^{-1}A} h'(\Phi[z]) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2} dz.
\end{aligned} \tag{11}$$

By definition, $\Phi^{-1}(A) = (-\infty, b]$, where $b = \Phi^{-1}(d)$. For every point x on the interval $(-\infty, b]$ we have $h'(\Phi[x]) \geq c$. Let us choose a subinterval $[a', b]$ of the interval $(-\infty, b]$. Since $h' \geq 0$ everywhere, we can assess that

$$\begin{aligned}
& \int_{-\infty}^{\infty} h'(\Phi[z]) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2} dz \\
&\geq c \int_{a'}^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2} dz = c[\Phi(b - \sigma\sqrt{t}) - \Phi(a' - \sigma\sqrt{t})],
\end{aligned} \tag{12}$$

where the last equation follows from integrating the density function of a normally distributed random variable with mean $\sigma\sqrt{t}$ and standard deviation 1 on the interval $[a', b]$.

Using the mean value theorem, for a $d \in [a', b]$ we assess that

$$c[\Phi(b - \sigma\sqrt{t}) - \Phi(a' - \sigma\sqrt{t})] \geq c(b - a')e^{-\frac{1}{2}(d - \sigma\sqrt{t})^2}.$$

Therefore, we can deduce that

$$\begin{aligned}
& e^{(\mu-r)t} \int_{-\infty}^{\infty} h'(\Phi[z]) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2} dz \\
&\geq e^{(\mu-r)t} c \cdot (b - a') e^{-\frac{1}{2}(d - \sigma\sqrt{t})^2} = c \cdot (b - a') e^{(\mu-r - \frac{\sigma^2}{2})t + d\sigma\sqrt{t} - \frac{1}{2}d^2}.
\end{aligned} \tag{13}$$

The last expression goes to infinity if t goes to infinity, since we assumed that $\mu - r - \frac{\sigma^2}{2} > 0$. It follows that $\rho_h(Y(t))$ in (9) goes to minus infinity and hence becomes negative for a large t . \square

4.2 Arbitrary Exponential Lévy Models

In this section, we assume that the stock process $S(t)$ is an exponential Lévy process. For a brief introduction, see Section 3. First, we derive an own formula for the spectral risk in this model.

Theorem 4.2. *In an arbitrary exponential Lévy model, the spectral risk of $Y(t)$,*

$$\rho_h(Y(t)) = S_0[1 - e^{(\mu-r)t} \int_{-\infty}^{\infty} e^{z-\kappa_X(1)t} h'(F_t(z)) f_t(z) dz], \tag{14}$$

where F_t is the distribution function of the Lévy process at a given time t , and f_t is its density function.

Proof. Using the Choquet representation in Definition 2.4, we get that

$$\rho_h(Y(t)) = \int_0^\infty h(1 - F_{Y(t)}(x))dx + \int_{-\infty}^0 [h(1 - F_{Y(t)}(x)) - 1]dx. \quad (15)$$

We conduct a substitution to express this formula with the distribution function of the Lévy process. To do so, we further shape $1 - F_{Y(t)}(x)$. We know that the log return of the stock in this model can be written as

$$\ln\left(\frac{S(t)}{S_0}\right) = \mu t + X_t - t\kappa_X. \quad (16)$$

Using Equation (16), we obtain the following

$$\begin{aligned} 1 - F_{Y(t)}(x) &= \mathbb{P}(S(t) < S_0 e^{rt} - x) \\ &= \mathbb{P}\left(X(t) < \ln\left(\frac{S_0 e^{rt} - x}{S_0}\right) - (\mu - \kappa_X(1))t\right) \\ &= F_t\left(\ln\left(\frac{S_0 e^{rt} - x}{S_0}\right) - (\mu - \kappa_X(1))t\right). \end{aligned} \quad (17)$$

In Equation (15), performing the substitution $z = \ln(e^{rt} - \frac{x}{S_0}) - (\mu - \kappa_X)t \Leftrightarrow dx = -S_0 e^{z+(\mu-\kappa_X)t} dz$, we get that

$$\rho_h(Y(t)) = \int_{-\infty}^C h(F_t(z)) S_0 e^{z+(\mu-\kappa_X)t} dz + \int_C^\infty [h(F_t(z)) - 1] S_0 e^{z+(\mu-\kappa_X)t} dz, \quad (18)$$

where $C = (r - \mu + \kappa_X)t$. Integrating by parts, we obtain that

$$\begin{aligned} \rho_h(Y(t)) &= [h(F_t(z)) S_0 e^{z+(\mu-\kappa_X)t}]_{-\infty}^C - \int_{-\infty}^C h'(F_t(z)) f_t(z) S_0 e^{z+(\mu-\kappa_X)t} dz \\ &\quad + [h(F_t(z) - 1) S_0 e^{z+(\mu-\kappa_X)t}]_C^\infty - \int_C^\infty h'(F_t(z)) f_t(z) S_0 e^{z+(\mu-\kappa_X)t} dz. \end{aligned} \quad (19)$$

Substituting the values on the boundary points using $h(0) = 0$, $h(1) = 1$ we can assert that

$$\rho_h(Y(t)) = S_0 e^{rt} [1 - e^{(\mu-r)t} \int_{-\infty}^\infty e^{z-\kappa_X t} h'(F_t(z)) f_t(z) dz].$$

□

Applying Theorem 4.2, we prove the same long-time behavior for arbitrary Lévy models as was proven by Nguyen, Pham, and Tran (2012) in the case of GBM.

Theorem 4.3. *In an arbitrary exponential Lévy model $\rho_h(Y(t))$ is negative for a large t if $h'(\cdot)$ is bounded below with a $c > 0$ constant, and if $\mu > r$.*

Proof. Due to the definition of κ_X and by the assumption of the theorem

$$\begin{aligned} &\int_{-\infty}^\infty e^{z-\kappa_X t} h'(F_t(z)) f_t(z) dz \\ &\geq C \cdot \int_{-\infty}^\infty e^{z-\kappa_X t} f_t(z) dz = C \cdot e^{-\kappa_X t} \mathbb{E}[e^{X_t}] = C. \end{aligned} \quad (20)$$

And therefore,

$$e^{(\mu-r)t} \int_{-\infty}^{\infty} e^{z-\kappa_X(1)t} h'(F_t(z)) f_t(z) dz \quad (21)$$

goes to infinity as t goes to infinity, implying that $\rho_h(Y(t))$ also goes to minus infinity and $\rho_h(Y(t))$ becomes negative for a large t . \square

4.3 FMLS Model

In this section, we assume that the stock process $S(t)$ is an FMLS process. For a brief introduction, see Section 3.

Theorem 4.4. *In the FMLS stock price model, the spectral risk of $Y(t)$,*

$$\rho_h(Y(t)) = S_0 [1 - e^{(\mu-r)t} \int_{-\infty}^{\infty} e^{z\sigma t^{1/\alpha} - \kappa t} h'(\Theta_\alpha(z)) \Theta'_\alpha(z) dz]. \quad (22)$$

Proof. The proof is similar to that of Theorem 4.2. The main difference appears in the variable substitution. Due to the properties of the stable variables (Proposition 3.1) we can express the risk without time dependency.

Performing the substitution $z = \frac{\ln(e^{rt} - \frac{x}{S_0}) - (\mu - \kappa)t}{\sigma t^{\frac{1}{\alpha}}} \Leftrightarrow dx = -S_0 \sigma t^{\frac{1}{\alpha}} e^{z\sigma t^{\frac{1}{\alpha}} + (\mu - \kappa)t} dz$, integrating by parts and utilising $h(0) = 0$ and $h(1) = 1$, we obtain $\rho_h(Y(t))$ as stated in the theorem. \square

Theorem 4.5. *In the FMLS stock price model, $\rho_h(Y(t))$ is negative for a large t if $h'(\cdot)$ is bounded below with a $c > 0$ constant, and if $\mu > r$.*

Proof. This is a special case of Theorem 4.3, since the FMLS process is an exponential Lévy process. \square

Theorem 4.6. *In the FMLS stock price model, $\rho_h(Y(t))$ is negative for a large t if $\mu > r + \kappa$.*

Proof. Since h' is a monotonically decreasing function in a closed interval, it can only have a countable number of jump points, so the function is almost surely continuous everywhere.

Let $A = [0, d]$. Due to the assumption of the theorem

$$h'(x) \geq c \quad \forall \quad x \in A,$$

where $c > 0$. As for the integral in (22), we obtain the following lower bound.

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{z\sigma t^{1/\alpha} - \kappa t} h'(\Theta_\alpha(z)) \Theta'_\alpha(z) dz \quad (23) \\ &= \int_{\Theta_\alpha^{-1}(A)} e^{z\sigma t^{1/\alpha} - \kappa t} h'(\Theta_\alpha(z)) \Theta'_\alpha(z) dz + \int_{R \setminus \Theta_\alpha^{-1}(A)} e^{z\sigma t^{1/\alpha} - \kappa t} h'(\Theta_\alpha(z)) \Theta'_\alpha(z) dz \\ &\geq c \int_{\Theta_\alpha^{-1}(A)} e^{z\sigma t^{1/\alpha} - \kappa t} \Theta'_\alpha(z) dz + \int_{R \setminus \Theta_\alpha^{-1}(A)} e^{z\sigma t^{1/\alpha} - \kappa t} h'(\Theta_\alpha(z)) \Theta'_\alpha(z) dz. \end{aligned}$$

By definition, $\Theta_\alpha^{-1}(A) = (-\infty, b]$, where $b = \Theta_\alpha^{-1}(a)$. For every point x on the interval $(-\infty, b]$, it holds that $h'(\Theta_\alpha(x)) \geq c$.

Next, we will show that $\Theta'_\alpha(z)$ is positive on an appropriate subinterval $[a', b']$ of $[-\infty, b]$. We know that $\Theta'_\alpha(z)$ is the density function of a stable distributed random variable with parameters $(\alpha, -1, 1, 0)$. Moreover, on $[-\infty, b]$ the cumulative distribution function of this random variable is not constant, hence the density function is surely positive at some points. According to Proposition 3.12 on page 102 of Cont and Tankov (2004), if a Lévy process is infinitely active, then the density function of this process is continuous for every t . Carr and Wu (2003) showed that the stable process is infinitely active in case $1 < \alpha < 2$, meaning that it has an infinite number of jumps over any time interval. Hence we can assess that the process $L_t^{\alpha, -1}$ has continuous density function for every $t \geq 0$. In particular, it holds for $t = 1$, that is for $\Theta'_\alpha(z)$. Thus $\Theta'_\alpha(z)$ is positive in at least one point on $[-\infty, b]$, implying that $\Theta'_\alpha(z)$ is positive on an appropriate subinterval $[a', b']$ of $[-\infty, b]$.

Denote the minimum value of $\Theta'_\alpha(z)$ on $[a', b']$ by $c' > 0$. Since $h' \geq 0$ everywhere, we obtain that

$$\begin{aligned} & e^{(\mu-r)t} \int_{-\infty}^{\infty} e^{z\sigma t^{1/\alpha} - \kappa t} h'(\Theta_\alpha(z)) \Theta'_\alpha(z) dz \\ & \geq c \cdot e^{(\mu-r)t} \int_{a'}^{b'} e^{z\sigma t^{1/\alpha} - \kappa t} \Theta'_\alpha(z) dz = \int_{a'}^{b'} c \cdot e^{(\mu-r-\kappa)t + z\sigma t^{1/\alpha}} \Theta'_\alpha(z) dz. \end{aligned} \quad (24)$$

Since $\Theta'_\alpha(z)$ is positive on an appropriate subinterval $[a', b']$ of $[-\infty, b]$, we obtain that

$$\begin{aligned} & e^{(\mu-r)t} \int_{-\infty}^{\infty} e^{z\sigma t^{1/\alpha} - \kappa t} h'(\Theta_\alpha(z)) \Theta'_\alpha(z) dz \\ & \geq c \cdot c' \int_{a'}^{b'} e^{(\mu-r-\kappa)t + z\sigma t^{1/\alpha}} dz. \end{aligned} \quad (25)$$

By integrating, it follows that

$$\begin{aligned} & \int_{a'}^{b'} e^{(\mu-r-\kappa)t + z\sigma t^{1/\alpha}} dz = \left[\frac{e^{(\mu-r-\kappa)t + z\sigma t^{1/\alpha}}}{\sigma t^{1/\alpha}} \right]_{a'}^{b'} \\ & = \frac{e^{(\mu-r-\kappa)t}}{\sigma t^{1/\alpha}} [e^{b'\sigma t^{1/\alpha} - a'\sigma t^{1/\alpha}}] = \frac{e^{(\mu-r-\kappa)t}}{\sigma t^{1/\alpha}} [e^{(b'-a')\sigma t^{1/\alpha}}]. \end{aligned} \quad (26)$$

The last expression goes to infinity if t goes to infinity, since we assumed that $\mu > r + \kappa$. It follows that $\rho_h(Y(t))$ in (22) goes to minus infinity and hence becomes negative for a large t . \square

5 Numerical Results

To illustrate the implications of our theoretical calculations, in this section we present numerical results performed with realistic model parameters. We obtained the daily closing prices on the dividend-adjusted SP500 index to represent the risky “stock” in our model. The cash deposit rate was estimated using data on the annualized returns on 3-month US treasury bills. Both time series were taken from 1960 to 2016. Calibrating the

historical data to the GBM model yielded the following estimation for the parameters: $r = 0.048$, $\mu = 0.089$, $\sigma_{GBM} = 0.155$.

To illustrate the effect of stronger tails in the loss distribution, and to offer some robustness test, we also calibrated a set of FMLS models with four different α parameters: $\alpha = 1.8, 1.85, 1.9, 1.95$. We used the same $r = 0.048$, $\mu = 0.089$ values as for the GBM model. At each α , we set σ_α such that the difference between the first and the third yearly return quartiles are matched with that of the GBM model. This procedure ensures consistency between the different models in terms of short-term volatility. All five of these models represent realistic test cases for long-term investments in a well-diversified US equity portfolio.

All the models are in the “high growth regime” as defined in Section 4. Using our results from Section 4, we computed the Expected Shortfall at the 90% level as an illustrative risk measure. We performed the calculation for all five calibrated models for different holding periods, up to one hundred years, with monthly steps. The value of the initial stock investment was set to unity. The distribution functions of the normal and stable distributions are readily available in MatLab, the necessary integrations were performed with numerical quadrature, also available in MatLab. The results are plotted in Figure 1. The risk curves start at zero, increase up to a maximum value, then, in accordance with our analytical results, drop down to and below zero. At any given holding period, the GBM model produces the lowest risk levels, while the FMLS curves, especially the ones with relatively low α parameters, show higher risk values. This is due to the heavier tails in the FMLS distribution on the loss side.

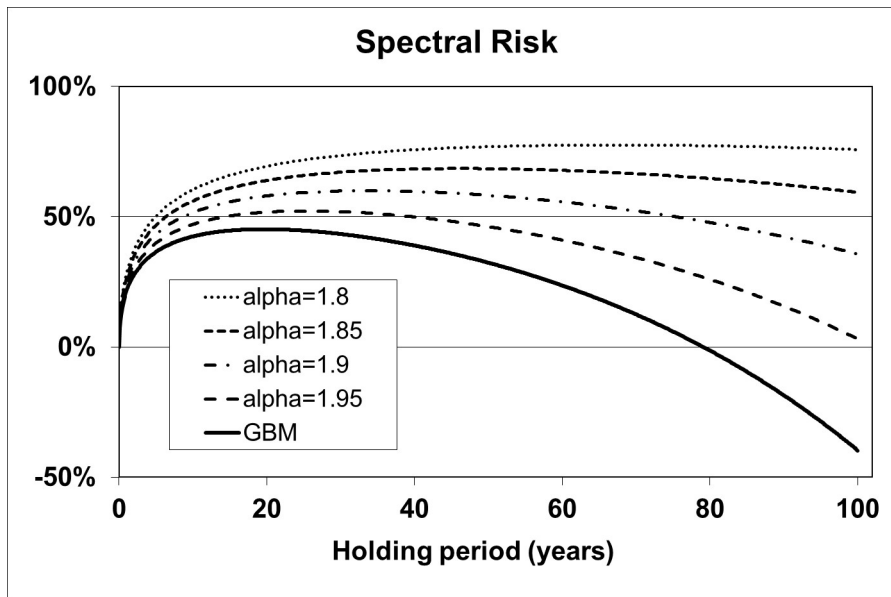


Figure 1: Expected Shortfall (90%) risk measure for holding a stock rather than a risk-free deposit, as a function of the holding period for $r = 0.048$, $\mu = 0.089$, and $\sigma_{GBM} = 0.155$. Risk curves are plotted for the GBM model, and for the FMLS model with different α parameters.

As discussed in the seminal paper of Artzner, Delbaen, Eber, and Heath (1999),

coherent risk measures can be interpreted as the amount of reserve cash necessary to lower market exposure to a tolerable level. Although the numerical results shown underscore our analytic conclusion that the risk measure becomes zero at some holding time for all the investigated models, note the time scale on the x axis. The necessary reserve cash becomes zero only at around, or beyond, one hundred years. Within our framework, it means that holding only stocks in our investment portfolio incurs a tolerable risk only if we are willing to hold them for more than a century. On the other hand, at long but realistic holding periods (a few decades), we need quite a sizable cushion to mitigate risk. According to Figure 1, the necessary cash reserve amounts to about half of the initial stock value.

To perform some sensitivity tests, let us consider the effect of changing the parameters. In particular, we will explore parameter modifications that lower the risk measure. For most users, decreasing the 90% confidence level for the Expected Shortfall is not an option. Increasing the confidence level would lead to higher risks, since then the average of even worse outcomes would be calculated. To decrease risk, volatility should be decreased or the difference between μ and r should be increased in our models. In Figure 2, we show risk curves with a reduced volatility of $\sigma_{GBM} = 0.125$. In Figure 3, the curves are calculated with the increased drift of $\mu = 0.11$. All other parameters are the same as in Figure 1. Our models with these modified parameters represent optimistic, but still realistic market views. Although the risk curves with the modified parameters shifted down as expected, our conclusions remain essentially the same: Risk remains significant for decades.

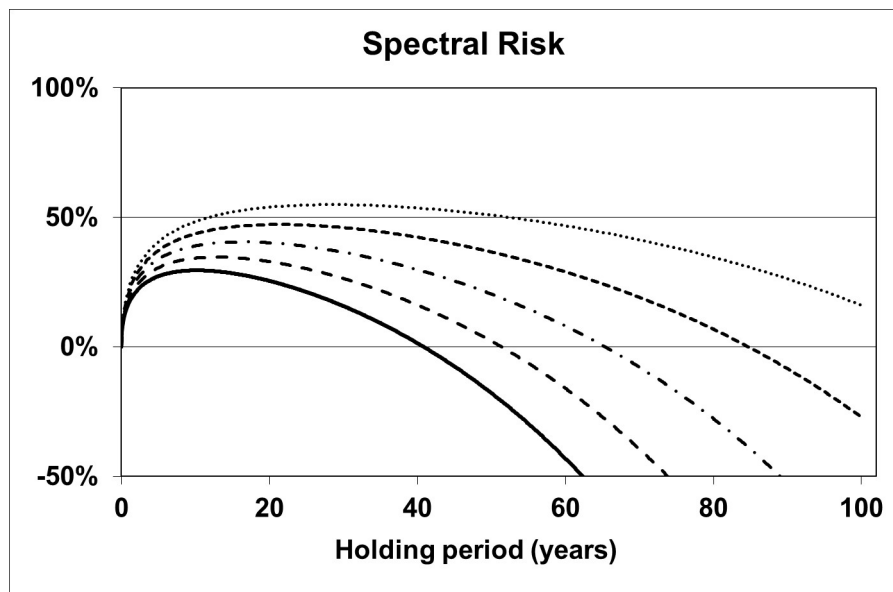


Figure 2: Expected Shortfall (90%) risk measure for holding a stock rather than a risk-free deposit, as a function of the holding period for $r = 0.048$, $\mu = 0.089$, and $\sigma_{GBM} = 0.125$. Risk curves are plotted for the GBM model, and for the FMLS model with different α parameters.

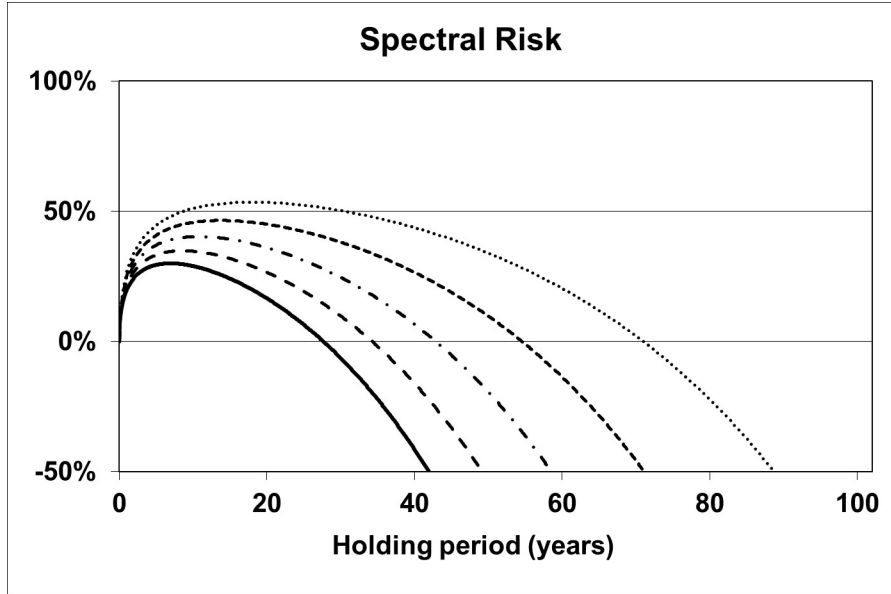


Figure 3: Expected Shortfall (90%) risk measure for holding a stock rather than a risk-free deposit, as a function of the holding period for $r = 0.048$, $\mu = 0.11$, and $\sigma_{GBM} = 0.155$. Risk curves are plotted for the GBM model, and for the FMLS model with different α parameters.

6 Conclusion

For long-run investors, it is crucial to know how to evaluate the risk associated with holding stocks rather than a risk-free deposit. We analyzed spectral risk, which can be interpreted as a weighted average regret value associated with having bought the stock, rather than investing in a money market account.

We show that for arbitrary exponential Lévy processes and when all possible outcomes have strictly positive weight, spectral risk becomes negative at long periods. We also prove the same behavior for all spectral risk measures (including the important special case of Expected Shortfall, giving zero weights to some good outcomes) when the stock price is in the high-growth regime and when it follows a GBM or an FMLS process. We leave it as an open question whether Expected Shortfall becomes negative for arbitrary exponential Lévy processes in the high growth regime.

After calibrating the model to SP500, we find that risk indeed becomes negative, but only about after a century, and it remains significant for decades. Increasing the tails in the FMLS process, or adding model or parameter uncertainty makes the situation even worse. Also supported by the sensitivity test, it is safe to conclude that holding stocks is risky in the long run.

References

ACERBI, C. (2002): “Spectral measures of risk: A coherent representation of subjective risk aversion,” *Journal of Banking & Finance*, 26, 1505–1518.

- ACERBI, C., AND G. SCANDOLO (2008): “Liquidity risk theory and coherent measures of risk,” *Quantitative Finance*, 8(7), 681–692.
- ACERBI, C., AND B. SZEKELY (2014): “Back-testing expected shortfall,” *Risk*, p. 76.
- ACERBI, C., AND D. TASCHE (2002): “Expected Shortfall: A Natural Coherent Alternative to Value at Risk,” *Economic Notes*, 31, 379–388.
- ARTZNER, P., F. DELBAEN, J. M. EBER, AND D. HEATH (1999): “Coherent measures of risk,” *Mathematical Finance*, 9, 203–228.
- AVRAMOV, D., S. CEDERBURG, AND K. LUČIVJANSKÁ (2017): “Are stocks riskier over the long run? Taking cues from economic theory,” *The Review of Financial Studies*, 31(2), 556–594.
- BASEL COMMITTEE (2013): “Fundamental review of the trading book: A revised market risk framework,” *Consultative Document*, October.
- BENNYHOFF, D. G. (2009): “Time diversification and horizon-based asset allocations,” *The Journal of Investing*, 18(1), 45–52.
- BERTOIN, J. (1996): *Lévy Processes*. Cambridge University Press.
- BODIE, Z. (1995): “On the risk of stocks in the long run,” *Financial Analysts Journal*, 51, 18–22.
- CARR, P., AND L. WU (2003): “The Finite Moment Log Stable Process and Option Pricing,” *The Journal of Finance*, LVIII, NO.2, 753–778.
- CHOQUET, G. (1954): “Theory of capacities,” in *Annales de l’institut Fourier*, vol. 5, pp. 131–295.
- CONT, R., AND P. TANKOV (2004): *Financial modelling with jump processes*. Chapman and Hall/CRC.
- CSÓKA, P., P. J.-J. HERINGS, AND L. Á. KÓCZY (2007): “Coherent measures of risk from a general equilibrium perspective,” *Journal of Banking & Finance*, 31(8), 2517–2534.
- FERGUSON, R., AND L. DEAN (1996): “On the risk of stocks in the long run: A Comment,” *Financial Analysts Journal*, 52, 67–68.
- HARLOW, W. V. (1991): “Asset allocation in a downside-risk framework,” *Financial Analysts Journal*, 47(5), 28–40.
- JANICKI, A., AND A. WERON (1994): *Simulation and Chaotic Behavior of Alpha-stable Stochastic Processes*. Marcel Dekker, New York.
- MANDELBROT, B. (1963): “New Methods in Statistical Economics,” *Journal of Political Economy*, 71, 421–440.

- MCCULLOCH, J. H. (1996): “Financial applications of stable distributions,” *Handbook of statistics*.
- NGUYEN, H. T., U. H. PHAM, AND H. D. TRAN (2012): “On some claims related to Choquet integral risk measures,” *Annals of Operations Research*, 195, 5–31.
- PÁSTOR, L., AND R. F. STAMBAUGH (2012): “Are stocks really less volatile in the long run?,” *The Journal of Finance*, 67(2), 431–478.
- ROCKAFELLAR, R. T., AND S. URYASEV (2002): “Conditional value-at-risk for general loss distributions,” *Journal of Banking & Finance*, 26(7), 1443–1471.
- SIEGEL, J. J. (1998): *Stocks for the Long Run*. New York: McGraw-Hill.
- SRIBOONCHITA, S., W.-K. WONG, S. DHOMPONGSA, AND H. T. NGUYEN (2009): *Stochastic dominance and applications to finance, risk and economics*. CRC Press.
- TREUSSARD, J. (2006): “The Non-Monotonicity of Value-at-Risk and the Validity of Risk Measures over Different Horizons,” available at <http://ssrn.com/abstract=776651>.
- WANG, S. (1996): “Premium calculation by transforming the layer premium density,” *ASTIN Bulletin: The Journal of the IAA*, 26(1), 71–92.
- WILKIE, D. (2001): “On the Risk of Stocks in the Long Run: a response to Zvi Bodie,” in *Proceedings of the 11th International AFIR Colloquium*, pp. 741–762.