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Riskiness, Risk Aversion, and Risk Sharing: Cooperation in a Dynamic Insurance Game

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Riskiness, Risk Aversion, and Risk Sharing:
Cooperation in a Dynamic Insurance Game

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Kockázatosság, kockázatkerülés és kockázatmegosztás: Együttműködés egy dinamikus biztosítási játékban

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Összefoglaló

Ez a tanulmány azt vizsgálja, hogy az együttműködés egy biztosítási játékban hogyan függ a kockázatkerülés mértékétől és a jövedelem kockázatosságától. Egy olyan dinamikus játékot vizsgálok, amelyet korlátotozott elköteleződés jellemez. Az együttműködés szintjét úgy definiálom, hogy az egyenlő a diszkontfaktorral, amely felett a tökéletes kockázatmegosztás önfenntartó. Amikor nincs aggregát kockázat, az együttműködés mértéke nagyobb, ha (i) a hasznosságfüggvény konkávabb, és ha (ii) a jövedelem kockázatosabb, ha a jövedelemeloszlás kockázatosságának kritériuma az átlagtartó spread, vagy a másodrendű sztochasztikus dominancia (SSD). Viszont ha a biztosírás nem teljes, (ii) nem mindig igaz, az egyéni és aggregát kockázat kölcsönhatása miatt. CARA (CRRA) preferenciák esetén az együttműködés pozitívan függ az abszolút (relatív) kockázatkerülési koefficienstől és a jövedelemeloszlás szórásától (relatív szórásától), és független az átlagjövedelemtől. Ez a tanulmány az együttműködés szintjét összefüggésbe hozza a biztosítási transzferekkel és a fogyasztás simaságával abban az esetben is, amikor a tökéletes kockázatmegosztás nem önfenntartó.

Tárgyszavak: informális biztosítás, korlátozott elköteleződés, kockázatkerülés, kockázatosság, komparatív statika, dinamikus sztochasztikus játékok

JEL: C73, D80

Riskiness, Risk Aversion, and Risk Sharing: Cooperation in a Dynamic Insurance Game*

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Abstract

This paper examines how cooperation in an insurance game depends on risk preferences and the riskiness of income. It considers a dynamic game where commitment is limited, and characterizes the level of cooperation as measured by the reciprocal of the discount factor above which perfect risk sharing is self-enforcing. When agents face no aggregate risk, there is more cooperation, if (i) the utility function is more concave, and if (ii) income is more risky considering a mean-preserving spread or an SSD deterioration. However, (ii) no longer holds when insurance can only be incomplete, because of the interplay of idiosyncratic and aggregate risk. In the case of exponential (isoelastic) utility, cooperation depends positively on both the coefficient of absolute (relative) risk aversion and the standard deviation (coefficient of variation), and is independent of mean income. This paper also relates the level of cooperation to informal insurance transfers and the smoothness of consumption when perfect risk sharing is not achieved.

Keywords: informal insurance, limited commitment, risk preferences, riskiness, comparative statics, dynamic stochastic games

JEL codes: C73, D80

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1 Introduction

Informal risk sharing occurs in a wide variety of economic contexts. Two neighbors in a village will help each other out, when one faces some negative shock, like illness, or crop loss due to pests. Members of a family also insure one another informally by, for example, helping out a member who becomes unemployed. Governments help one another in case of a natural disaster, or a currency crisis. An employer and her employee insure each other against the fluctuations of the market wage. However, while cooperation to share risk has obvious benefits in the long run, in all these cases, the agent who gets the positive shock today has an incentive to walk away from the informal insurance arrangement. This paper examines how cooperation in this context is determined by agents' risk aversion and the riskiness of the environment.

Informal insurance is modeled by risk sharing with limited commitment (Thomas and Worrall, 1988; Kocherlakota, 1996; Ligon, Thomas, and Worrall, 2002). The idea is that agents may enter into a risk sharing arrangement to mitigate the adverse effects of risk they face, even when formal insurance contracts are not available. In particular, let us assume that agents' only possibility to smooth consumption across states of the world is to play the "informal insurance game" (Coate and Ravallion, 1993). This involves deciding on an insurance transfer once incomes are realized. The transfers have to be voluntary, or, self-enforcing. That is, in every period and state of the world, each agent may renege on the contract, and consume her own income in every subsequent period. What makes transfers possible today is the expected gain from future insurance. Full cooperation means that agents achieve perfect risk sharing.

How can we relate cooperation to the discount factor? On the one hand, if an agent has a high preference for the present, that is, a low discount factor, she will be less willing to make a transfer today. On the other hand, as the discount factor approaches 1, perfect risk sharing, the first best, becomes self-enforcing, according to the well-known folk theorem result. Thus, when the discount factor is sufficiently high, full cooperation occurs. How high it has to be will depend on risk preferences and the riskiness of the distribution that yields income.

After setting up the model, section 2 shows how to find the level of cooperation, that is defined as the reciprocal of the discount factor above which perfect risk sharing is self-enforcing. Intuitively, it will be determined by the trade-off between the expected future gains of insurance and the utility cost of making a transfer today.

Afterwards, this paper looks at some comparative statics related to risk aversion and riskiness. First, section 3 examines the two most widely-used preference classes, namely, utility functions characterized by constant absolute risk aversion (CARA), and those characterized by constant relative risk aversion (CRRA). Then, I examine general, increasing and concave utility functions. Two scenarios are considered. The first case, examined in section 4, is when perfect risk sharing results in completely smooth consumption across states and time. This is equivalent to there being no aggregate uncertainty. Section 5 deals with the second case, when agents still suffer from consumption fluctuations, even though they share risk perfectly. This may be thought of as the case with aggregate uncertainty.

Section 6 discusses how the level of cooperation is related to the solution of the risk sharing with limited commitment model. It shows, by way of a numerical example, that, if the level of cooperation is higher in some environment, characterized by the risk preferences and the distribution of income, then insurance transfers are higher and consumption is smoother for any discount factor such that some but not perfect risk sharing occurs.

Kimball (1988) was the first to argue that informal risk sharing in a community may be achieved with voluntary participation of all members. His computations for the constant relative risk aversion (CRRA) case also suggest that risk sharing arrangements are less likely to exist, the lower the discount factor. Thomas and Worrall (1988) build a model of two-sided limited commitment in a dynamic wage contract setting. Early contributions to modeling risk sharing with limited commitment include Coate and Ravallion (1993), who introduce two-sided limited commitment in a dynamic model, but they restrict contracts to be static. Fafchamps and Lund (2003) argue that enforcement constraints play an important role in informal insurance arrangements, based on evidence from rural Philippines.

Kocherlakota (1996) allows for dynamic contracts, and proves existence and some important properties of the solution. Ligon, Thomas, and Worrall (2002) characterize and calculate the solution of the model of risk sharing with limited commitment. They prove that there exists a discount factor above which perfect risk sharing is self-enforcing, and there also exists a discount factor below which agents stay in autarky. Genicot and Ray (2002) give a sufficient condition for nontrivial risk sharing contracts to exist. This paper deals with the other threshold.

Genicot (2006) examines how the likelihood of perfect risk sharing, defined as 1 minus the discount factor above which perfect risk sharing is self enforcing, changes with wealth inequality, in the case where preferences are characterized by hyperbolic absolute risk aversion (HARA). Dubois (2006) considers quadratic utility, and shows that the value of perfect risk sharing relative to autarky is increasing in risk aversion. Krueger and Perri (2006) argue that more cross-sectional income inequality leads to more insurance, thus cross-sectional consumption inequality increases less, or may even decrease. More cross-sectional income inequality is in fact equivalent to more volatile income. Fafchamps (1999) shows that in the case of a static contract, under some conditions, one can always find a concave transformation of the utility function, or a mean-preserving spread, that destroys the sustainability of the risk sharing arrangement (see Fafchamps, 1999, proposition 3). This paper establishes conditions under which the desirable comparative static results hold for the prefect risk sharing contract to be sustainable, or, self-enforcing.

The rest of the paper is structured as follows. Section 2 presents a model of risk sharing with limited commitment, and shows how to determine the reciprocal of the discount factor above which perfect risk sharing is self-enforcing. Section 3 to 5 present the comparative statics related to risk aversion and riskiness. Section 6 discusses how to measure informal insurance when full cooperation is not possible. Section 7 concludes.

2 The level of cooperation in an insurance game

This section first sets up the model of informal insurance. In particular, I use a model of risk sharing with limited commitment, following Thomas and Worrall (1988), Kocherlakota (1996), Ligon, Thomas, and Worrall (2002), and others. The model has a wide range of interpretations. One may have in mind households in a village (Ligon, Thomas, and Worrall, 2002; Attanasio and Ríos-Rull, 2000), members of a family (Mazzocco, 2007), an employee and an employer (Thomas and Worrall, 1988), or countries (Kehoe and Perri, 2002). Further, Schechter (2007) uses the same model to examine the interaction between a farmer and a thief, and Dixit, Grossman, and Gul (2000) use a similar model to examine cooperation between opposing political parties[1].

Afterwards, section 2.2 shows how to find the discount factor above which full cooperation, or, perfect risk sharing, occurs. The *level of cooperation* is then defined as its reciprocal.

2.1 Modeling informal insurance

Consider an economy with two infinitely-lived, risk-averse agents[2], who receive a stochastic endowment, or income, each period. Note that, in this paper, income is the sum of any exogenous revenue, plus the payoff from any gamble "played". Note that risk is exogenous, agents cannot choose not to play the gamble.

Suppose that income of both agents follows the same discrete distribution, Y, with positive and finite possible realizations, and is independently and identically distributed (i.i.d.) across time periods[3]. Let s_t (lower index t) denote the state of the world realized, and $y_i(s_t)$ the income realization for agent i at state s and time t. Let $s^t = (s_1, s_2, ..., s_{t-1}, s_t)$ (upper index t) denote the history of income states up to t. Consumption smoothing across states of nature, and not across time is considered here, and I assume that no savings, or storage is possible. Further, agents hold the same beliefs about the income processes ex ante, and income realizations are common knowledge ex post.

[1] I thank Refik Emre Aytimur for this reference.
[2] The model can easily be extended to n agents.
[3] The model can be extended to the case where income follows a Markov-chain.

Denote the utility function by $u()$, defined over a single, private, and perishable consumption good c. Suppose that $u()$ is strictly increasing, twice continuously differentiable, strictly concave, so agents are risk averse, and egoistic in the sense that agents only care about their own consumption. Each agent $i \in \{1, 2\}$ maximizes her expected lifetime utility,

$$E_0 \sum_t \delta^t u\left(c_i\left(s^t\right)\right), \tag{1}$$

where E_0 is the expected value at time 0 calculated with respect to the probability measure describing the common beliefs, $\delta \in (0, 1)$ is the (common) discount factor, and $c_i\left(s^t\right)$ is consumption of agent i when history s^t has occurred. While income is i.i.d., the consumption allocation may depend on the whole history of income realizations, s^t. Note also that agents are supposed ex-ante identical, that is, they have the same preferences and both get their endowment as a realization of Y, they differ only in their income realizations. This assumption is useful when we want establish comparative static results with respect to risk aversion and the riskiness of the distribution that yields income.

To attenuate the adverse effects of the risk they face, agents may enter into an informal risk sharing arrangement. In particular, they play the following dynamic informal insurance game (Coate and Ravallion, 1993). At each $t \in \{1, 2, ...\}$, the state of the world, say $\widetilde{s}$, is realized. Incomes are given by $\{y_{it}(\widetilde{s})\}_i$. Then, each agent may transfer some amount $\tau_{it}(\widetilde{s})$ to her risk sharing partner. Finally, consumption takes place, in particular, $c_{it}(\widetilde{s}) = y_{it}(\widetilde{s}) - \tau_{it}(\widetilde{s}) + \tau_{-it}(\widetilde{s})$, $\forall i$, where $-i$ denotes the other agent. We will characterize the equilibrium is terms of the consumption allocation $\{c_{it}(s)\}_i$.

We are looking for the subgame-perfect Nash equilibrium (SPNE) that is constrained Pareto optimal. Note that both agents staying in autarky in each period is a SPNE that requires no cooperation. Thus each agent has to be at least as well off respecting the terms of the informal risk sharing contract, as consuming her own income today and in all subsequent periods, at each history s^t. Moreover, the trigger strategy of reverting to autarky is the most severe subgame-perfect punishment in this context (Ligon, Thomas, and Worrall, 2002). In other words, it is an optimal penal code in the sense of Abreu (1988). The trigger strategy can be thought of as a breakdown of trust, that is, once an agent failed to help out her risk sharing partner, the later is not be willing to enter into any informal insurance arrangement with her anymore.

One may write the problem as follows. The (utilitarian) social planner maximizes a weighted sum of agents' utilities,

$$\max_{\{c_i(s^t)\}} \sum_{t=1}^{\infty} \sum_{s^t} \delta^t Pr\left(s^t\right) u\left(c_1\left(s^t\right)\right) + x_0 \sum_{t=1}^{\infty} \sum_{s^t} \delta^t \Pr\left(s^t\right) u\left(c_2\left(s^t\right)\right), \tag{2}$$

where $Pr\left(s^t\right)$ is the probability of history s^t occurring, and x_0 is the (initial) relative weight of agent 2 in the social planner's objective; subject to the resource constraints,

$$\sum_i c_i\left(s^t\right) \leq \sum_i y_i\left(s_t\right), \forall s^t, \tag{3}$$

and the enforcement constraints,

$$\sum_{r=t}^{\infty} \sum_{s^r} \delta^{r-t} Pr\left(s^r\right) u\left(c_i\left(s^r\right)\right) \geq U_i^{aut}\left(s_t\right), \forall s_t, \forall t, \tag{4}$$

where $U_i^{aut}\left(s_t\right)$ is the expected lifetime utility in autarky when state s_t has occurred today.

Denoting the Lagrange multipliers on the enforcement constraints (4) by $\delta^t \mu_i(s_t)$, and introducing the co-state variable,

$$x(s^t) \equiv \frac{x_0 + \mu_2(s_1) + \mu_2(s_2) + ... + \mu_2(s_t)}{1 + \mu_1(s_1) + \mu_1(s_2) + ... + \mu_1(s_t)},$$

we may rewrite the problem in a recursive form (Marcet and Marimon, 1998). In particular, the value function can be written as

$$V_i\left(s_t, x_{t-1}\right) = u\left(c_i\left(s_t, x_{t-1}\right)\right) + \delta \sum_{s_{t+1}} Pr\left(s_{t+1}\right) V_i\left(s_{t+1}, x_t\left(s_t, x_{t-1}\right)\right),$$

where x_t is the ratio of marginal utilities, or, the relative weight of agent 2 at time t. Numerical dynamic programming can be used to solve for the function $x_t\left(s_t, x_{t-1}\right)$ that fully characterizes the solution[4] Once we know x_t, the consumption allocation can easily be found using the first order conditions with respect to consumption, and the resource constraint.

In terms of the patter of binding enforcement constraints, three cases are possible. Given the utility functions, the discount factor, and the distribution of income, Y, at the constrained-efficient solution there might be (i) no risk sharing, that is, agents stay in autarky, (ii) perfect

[4]See Laczo, 2008, for an algorithm in a more general case when the income state follows a Markov process.

risk sharing, or (iii) something in between, that is, partial insurance. Let us look at each case in turn.

First, in autarky the agents' maximization problem is trivial, since resources are not transferable across time. Each agent consumes her own income in each state and time period. Since we have supposed that the income process is i.i.d., the expected lifetime utility of agent i, at state $\widetilde{s}$, for all t, can be written as

$$u\left(y_i\left(\widetilde{s}\right)\right) + \frac{\delta}{1-\delta}\sum_s Pr\left(s\right)u\left(y_i\left(s\right)\right). \tag{5}$$

Note that, by definition, the informal risk sharing contact must provide at least the lifetime utility (5), in each state $\widetilde{s}$ and at each time t, for agents to voluntary participate.

Second, in the case of perfect risk sharing, all idiosyncratic risk is eliminated. To find the perfect risk sharing solution, or, the set of Pareto-optimal allocations, one may consider the social planner's problem above, but without the enforcement constraints. The first order conditions yield the standard result that

$$\frac{u'\left(c_1\left(s_t\right)\right)}{u'\left(c_2\left(s_t\right)\right)} = x_0, \forall s, \forall t, \tag{6}$$

that is, the ratio of marginal utilities is constant across time and states of nature in the case of perfect risk sharing. Replacing for $c_2\left(s_t\right)$ in (6) using the resource constraint (3), the consumption allocation can be easily solved for. Let $c_1^*\left(s_t, x_0\right)$ and $c_2^*\left(s_t, x_0\right)$ denote the solution, in other words, the sharing rule. Taking into account that income is distributed according to Y in each period, the expected lifetime utility for agent i at state $\widetilde{s}$, can be written as

$$u\left(c_i^*\left(\widetilde{s}, x_0\right)\right) + \frac{\delta}{1-\delta}\sum_s Pr\left(s\right)u\left(c_i^*\left(s, x_0\right)\right). \tag{7}$$

Note that the consumption allocation only depends on aggregate income[5] and the relative weight of agent 2 in the social planner's objective. Thus we may also write the sharing rule as $c_1^*\left(y_1\left(s\right) + y_2\left(s\right), x_0\right)$ and $c_2^*\left(y_1\left(s\right) + y_2\left(s\right), x_0\right)$.

The third case is when some, but not perfect insurance is achieved. This case is often referred to as partial insurance. Here the perfect risk sharing solution is not self-enforcing,

[5]This property is sometimes referred to as income pooling, or as the mutuality principle.

there is at least one enforcement constraint that binds. In other words, full cooperation is not possible. This means that at some state $\widetilde{s}$ for one of the agents, the lifetime utility from perfect risk sharing (7) would be smaller than the autarky utility (5). In such states the informal risk sharing contract will determine an allocation such that this agent is indifferent between respecting the terms of the contract or deviating to autarky.

The solution can be fully characterized by the function $x_t(s_t, x_{t-1})$, and in particular, by a set of state-dependent intervals on the relative weight of household 2, or, the ratio of marginal utilities, x, that give the possible relative weights in each income state. Denote the interval for state s by $[\underline{x}^s, \overline{x}^s]$. Suppose that last period the ratio of marginal utilities was x_{t-1}, and today the income state is s. Today's ratio of marginal utilities, x_t, is determined by the following updating rule (Ligon, Thomas, and Worrall, 2002):

$$x_t = \begin{cases} \overline{x}^s & \text{if} \quad x_{t-1} > \overline{x}^s \\ x_{t-1} & \text{if} \quad x_{t-1} \in [\underline{x}^s, \overline{x}^s] \\ \underline{x}^s & \text{if} \quad x_{t-1} < \underline{x}^s \end{cases}$$

When an enforcement constraint binds, we cannot keep x constant (as in the perfect risk sharing case). However, intuitively, we will try to keep x_t as close as possible to x_{t-1}. The constrained-efficient solution has a number of interesting properties, including history dependence, and a quasi-credit element (Fafchamps, 1999). Details are given in Kocherlakota (1996), Ligon, Thomas, and Worrall (2002), and Laczó (2008), among others. Note that, in the case when perfect risk sharing is self-enforcing, all the $[\underline{x}^s, \overline{x}^s]$ overlap, that is, there exists some $\tilde{x}$ such that $\tilde{x} \in [\underline{x}^s, \overline{x}^s]$, $\forall s$. Further, if such an $\tilde{x}$ exists, then it will be reached with probability 1 after a sufficient number of periods (Kocherlakota, 1996).

Let us examine how the above three cases evolve as the discount factor, δ, changes. Take risk preferences and the income processes given. For δ approaching zero, the agent receiving high income today will not make a transfer, since she values current consumption too much. Thus, for low values of δ, we are in the autarky case. On the other extreme, according to the well-known folk theorem result, the first best is achieved for a discount factor sufficiently close to 1. Finally, for some intermediate values of δ, partial insurance occurs, that is, we are in the third case. For the purposes of this paper, what is important is the following: there exists a level of the discount factor, given preferences and the income process, above which perfect risk sharing is self-enforcing (Ligon, Thomas, and Worrall, 2002, proposition 2, part (ii)). Denote this discount factor by δ^*.

2.2 Determining the level of cooperation

Now I show how to find δ^*, the discount factor such that, for all $\delta \geq \delta^*$, perfect risk sharing is self-enforcing. In other words, we are are looking for the lowest possible discount factor such that (i) perfect risk sharing occurs, that is, the ratio of marginal utilities is constant across states and over time, denoted x^*, and (ii) the enforcement constraints are satisfied. In mathematical terms, there exists x^* such that (7), with $x^* = x_0$, is greater than (5), for all $\widetilde{s}$.

Intuitively, an enforcement constraint is most stringent when agent 1 has the highest possible income realization, y^h, while the agent 2 has the lowest possible one, y^l, or the reverse. Let us denote these states by hl and lh, respectively. This is when the autarky lifetime utility is highest, and when the biggest transfer should be made to respect the terms of the perfect risk sharing contract.

The expected lifetime utility of agent 1 in autarky, when her income realization is y^h today, is

$$u\left(y^h\right) + \frac{\delta}{1-\delta} \sum_s Pr\left(s\right) u\left(y_1\left(s\right)\right), \tag{8}$$

while for agent 2 it is

$$u\left(y^h\right) + \frac{\delta}{1-\delta} \sum_s Pr\left(s\right) u\left(y_2\left(s\right)\right). \tag{9}$$

Since agents are assumed ex-ante identical, that is, they have the same preferences and their income is generated from the same random variable Y, (8) and (9) are equal, and can be written as

$$u\left(y^h\right) + \frac{\delta}{1-\delta} \sum_s Pr\left(s\right) u\left(y_i\left(s\right)\right) \equiv u\left(y^h\right) + \frac{\delta}{1-\delta} Eu(y), \tag{10}$$

where $Eu(y)$ is the expected per-period utility in autarky.

The expected lifetime utility of agent 1 in the perfect risk sharing case, when she is earning

y^h and agent 2 is getting y^l today, is

$$u\left(c_1^*\left(y^h+y^l,x^*\right)\right)+\frac{\delta}{1-\delta}\sum_s Pr\left(s\right)u\left(c_1^*\left(y_1\left(s\right)+y_2\left(s\right),x^*\right)\right).$$

This expression is the same as (7) with $x^*=x_0$, and making explicit that the consumption allocation depends on state s only through aggregate income. Similarly, the value of perfect risk sharing for agent 2, when her income is y^h and agent 1 is earning y^l, is

$$u\left(c_2^*\left(y^h+y^l,x^*\right)\right)+\frac{\delta}{1-\delta}\sum_s Pr\left(s\right)u\left(c_2^*\left(y_1\left(s\right)+y_2\left(s\right),x^*\right)\right).$$

One can find $c_1^*\left(y_1\left(s\right)+y_2\left(s\right),x^*\right)$ and $c_2^*\left(y_1\left(s\right)+y_2\left(s\right),x^*\right)$ using the first order conditions, equation (6).

We are looking for the lowest possible discount factor such that the following two enforcement constraints are satisfied:

$$u\left(y^h\right)+\frac{\delta}{1-\delta}Eu(y)\leq u\left(c_1^*\left(y^h+y^l,x^*\right)\right)+\frac{\delta}{1-\delta}\sum_s Pr\left(s\right)u\left(c_1^*\left(y_1\left(s\right)+y_2\left(s\right),x^*\right)\right)\quad(11)$$

and

$$u\left(y^h\right)+\frac{\delta}{1-\delta}Eu(y)\leq u\left(c_2^*\left(y^h+y^l,x^*\right)\right)+\frac{\delta}{1-\delta}\sum_s Pr\left(s\right)u\left(c_2^*\left(y_1\left(s\right)+y_2\left(s\right),x^*\right)\right),\quad(12)$$

where x^* is the ratio of marginal utilities, or, the relative weight of agent 2 in the social planner objective, that is reached after a sufficient number of periods with probability 1, starting from any initial relative weight x_0 (see Kocherlakota, 1996). Remember that, if the enforcement constraints (11) and (12), relating to the most unequal states hl and lh, respectively, are satisfied, then the enforcement constraints of all other states will be satisfied as well. Using the following lemma, finding δ^* will be easy.

Lemma 1. $x^*=1$. *Equivalently, agents consume the same amount, in other words, aggregate income is shared equally, when the discount factor equals* δ^*, *that is*

$$c_i^*(s,x^*)=c_{-i}^*(s,x^*)=\frac{y_i(s)+y_{-i}(s)}{2},\forall s,\forall t.$$

Proof. Let us first distinguish three cases concerning the consumption allocation according to the value of x^*.

- $x^* = 1$. Then, from (6), $u'(c_1^*(y_1(s)+y_2(s), x^*)) = u'(c_2^*(y_1(s)+y_2(s), x^*))$. It follows immediately that $c_1^*(y_1(s)+y_2(s), x^*) = c_2^*(y_1(s)+y_2(s), x^*) = \frac{y_1(s)+y_2(s)}{2}$, $\forall s$.

- $x^* > 1$. Then $u'(c_1^*(y_1(s)+y_2(s), x^*)) > u'(c_2^*(y_1(s)+y_2(s), x^*))$, and $c_1^*(y_1(s)+y_2(s), x^*) < c_2^*(y_1(s)+y_2(s), x^*)$, since $u'()$ is decreasing. Thus in this case $c_1^*(y_1(s)+y_2(s), x^*) < \frac{y_1(s)+y_2(s)}{2}$ and $c_2^*(y_1(s)+y_2(s), x^*) > \frac{y_1(s)+y_2(s)}{2}$,$\forall s$.

- $x^* < 1$. Similarly, $c_1^*(y_1(s)+y_2(s), x^*) > \frac{y_1(s)+y_2(s)}{2}$ and $c_2^*(y_1(s)+y_2(s), x^*) < \frac{y_1(s)+y_2(s)}{2}$,$\forall s$.

The proof is by contradiction. Suppose that $x^* \neq 1$, and, without loss of generality, assume further that $x^* > 1$. First, note that $u(y^h) > u(c_i^*(y^h+y^l, x^*))$ but $Eu(y) < \sum_s Pr(s)\, u(c_i^*(y_1(s)+y_2(s), x^*))$, $\forall i$, thus the constraints (11) and (12) are more stringent for a lower δ. Therefore, minimizing δ, at least one of the two constraints must hold with equality. Let us consider the two cases in turn.

- (11) holds with equality. We have seen above that for $x^* > 1$, $c_1^*(y_1(s)+y_2(s), x^*) < \frac{y_1(s)+y_2(s)}{2} < c_2^*(y_1(s)+y_2(s), x^*)$, $\forall s$, thus (12) is slack. Then, (11) can be used to solve for δ^*. Rearranging (11) gives

$$\delta^* = \frac{u(y^h) - u(c_1^*(y^h+y^l, x^*))}{u(y^h) - u(c_1^*(y^h+y^l, x^*)) + \sum_s Pr(s)\, u(c_1^*(y_1(s)+y_2(s), x^*)) - Eu(y)}. \quad (13)$$

  Now, consider the following alternative allocation. Transfer a small amount $\epsilon(s)$ from agent 2 to agent 1 at state s, $\forall s$, such that (12) still holds. As a result, δ^* given in (13) decreases, because the term $u(y^h) - u(c_1^*(y^h+y^l, x^*))$ decreases, while the term $\sum_s Pr(s)\, u(c_1^*(y_1(s)+y_2(s), x^*)) - Eu(y)$ increases. Thus the original solution cannot be the one corresponding to the lowest δ.

- (12) holds with equality. In this case, (11) is violated, since $c_1^*(y_1(s)+y_2(s), x^*) < \frac{y_1(s)+y_2(s)}{2} < c_2^*(y_1(s)+y_2(s), x^*)$, $\forall s$.

Thus x^* cannot be different from 1, as I wanted to show. $\square$

Then, the expected lifetime utility of perfect risk sharing for agent i, in the state when she is getting y^h and agent $-i$ is earning y^l, can be written as

$$u\left(\frac{y^h+y^l}{2}\right)+\frac{\delta}{1-\delta}\sum_s Pr\left(s\right)u\left(\frac{y_i(s)+y_{-i}(s)}{2}\right). \tag{14}$$

Now we are ready to determine δ^* explicitly as a function of the the distribution that yields income and the utility function $u()$. Proposition 1 shows the formula.

Proposition 1. *The discount factor above which perfect risk sharing is self-enforcing, δ^*, is given by*

$$\delta^*=\frac{u\left(y^h\right)-u\left(\frac{y^h+y^l}{2}\right)}{u\left(y^h\right)-u\left(\frac{y^h+y^l}{2}\right)+\sum_s Pr\left(s\right)\left[u\left(\frac{y_i(s)+y_{-i}(s)}{2}\right)-u\left(y_i\left(s\right)\right)\right]}.$$

Proof. Equating (10) and (14), and rearranging yields the result. □

Note that a lower δ^* means that cooperation is possible for a wider range of discount factors. Thus I define its reciprocal, $1/\delta^*$, as the level of cooperation.

Definition 1. *I call the reciprocal of the discount factor above which perfect risk sharing is self-enforcing the level of cooperation. It is given by*

$$\frac{1}{\delta^*}=1+\frac{\sum_s Pr\left(s\right)\left[u\left(\frac{y_i(s)+y_{-i}(s)}{2}\right)-u\left(y_i\left(s\right)\right)\right]}{u\left(y^h\right)-u\left(\frac{y^h+y^l}{2}\right)}.$$

Since agents are ex-ante identical, if there is a state s occurring with probability $Pr\left(s\right)$, where agent i is earning $y_i\left(s\right)$ and agent $-i$ is getting $y_{-i}\left(s\right)$, then there is also a state, denoted $-s$, occurring with probability $Pr\left(-s\right)=Pr\left(s\right)$, with agent i receiving $y_i\left(-s\right)=y_{-i}\left(s\right)$ and agent $-i$ getting $y_{-i}\left(-s\right)=y_i\left(s\right)$. Therefore one may also write the level of cooperation as

$$\frac{1}{\delta^*}=1+\frac{\sum_s Pr\left(s\right)\left[u\left(\frac{y_i(s)+y_i(-s)}{2}\right)-\frac{1}{2}\left(u\left(y_i\left(s\right)\right)+u\left(y_i\left(-s\right)\right)\right)\right]}{u\left(y^h\right)-u\left(\frac{y^h+y^l}{2}\right)}.$$

Let $\overline{y}$ denote per-capita income in the extreme states, hl and lh, that is, $\overline{y}=\frac{y^h+y^l}{2}$, and note that $E\left(y\left(s\right)\right)=\frac{1}{2}\left(u\left(y_i\left(s\right)\right)+u\left(y_i\left(-s\right)\right)\right)$ is consumption in states s and $-s$ when agents share risk perfectly. Then,

$$\frac{1}{\delta^*}=1+\frac{\sum_s Pr\left(s\right)u\left[E\left(y\left(s\right)\right)\right]-\sum_s Pr\left(s\right)u\left(y_i\left(s\right)\right)}{u\left(y^h\right)-u\left(\overline{y}\right)} \tag{15}$$

The second term on the right hand side is positive, because both the numerator and the denominator are positive for $u()$ increasing and strictly concave. It follows that $1/\delta^* > 1$, thus $\delta^* < 1$. δ^* is also positive, given that income realizations are bounded.

The numerator and the denominator on the right hand side of (15) have natural interpretations. The numerator is the expected future (one-period) gain of sharing risk perfectly rather than staying in autarky. The denominator is today's cost of respecting the terms of the risk sharing contract, at the state where the agent is earning y^h, while her risk sharing partner is getting y^l, that is, when respecting the contract is most costly. Using δ^* to discount future net benefits, they should be just important enough to compensate the agent for the loss she incurs today by making the transfer $y^h - \left(y^h + y^l\right)/2$. Thus, full cooperation occurs when the discount factor is higher than the threshold δ^*, while it is not sustainable if the discount factor is lower.

The next two sections examine how the $1/\delta^*$ is related to risk preferences and the riskiness of the income distribution, Y. First, one would like to say that, if agents are more risk averse, more cooperation is possible in the insurance game. Just as in the standard setting with a risk-averse agent purchasing insurance from an insurance company, one would like to have that a more risk-averse agent is willing to pay more to avoid a given risk. Second, similarly, if income is more risky, agents have more incentive to cooperate, just as they are expected to be willing to pay more for formal insurance. First, we look at the special cases of CARA and CRRA preferences. Then, in section 4 and 5, we turn to the general case of any increasing and concave utility function.

3 CARA and CRRA preferences

This section performs comparative static exercises for the two most widely used utility functions. Namely, preferences characterized by constant absolute risk aversion (CARA) and constant relative risk aversion (CRRA) are examined in turn. I consider a simple setting with only two income realizations, y^h and y^l. Let us denote by Pr^{asym} the probability of the asymmetric states, hl and lh, occurring. Remember that $\overline{y} = \frac{y^h+y^l}{2}$. In this case, the level of cooperation simplifies to

$$\frac{1}{\delta^*} = 1 - Pr^{asym} + Pr^{asym} \frac{u\left(\overline{y}\right) - u\left(y^l\right)}{u\left(y^h\right) - u\left(\overline{y}\right)} \tag{16}$$

Extending the results to more income states is left for future work. The aim of this section is to show that, considering standard examples of parametrized utility functions, we have the desired comparative static results for $1/\delta^*$. I also investigate what the appropriate measure of riskiness is in these cases, relating riskiness to informal insurance.

3.1 CARA preferences

Suppose that the utility function $u()$ takes the standard exponential form. In mathematical terms,

$$u\left(c\right) = -\frac{1}{A} exp\left(-Ac\right), \tag{17}$$

where $A > 0$ is the coefficient of absolute risk aversion. We know that, if a CARA agent, with given A and wealth, is indifferent between accepting or not accepting a gamble with given mean and standard deviation (an additive risk), then this is true for any wealth level. Further, the variance or standard deviation of a stochastic process is often used as a simple measure of risk.

Let $y^h = y$ and $y^l = y - p$, with $y > 0$ and $y > p > 0$. So mean income is $y - \frac{p}{2}$, and the standard deviation is $\frac{p}{2}$. Notice that the standard deviation only depends on p and is independent of y, that I will call the level of incomes. Further, any mean - standard deviation combination can be reproduced by choosing p and y appropriately.

First of all, it is of interest to see what parameters of the model determine $1/\delta^*$, the level of informal insurance. Then, we will examine the relationship between the level of cooperation, and (i) risk aversion as measured by A, and (ii) the riskiness of income as measured by p, or, the standard deviation. We expect $1/\delta^*$ to increase with both A and p. This is indeed the case, as Claim 1 states.

Claim 1. *In the CARA case, $1/\delta^*$ depends positively on A and p, and is independent of y. That is, cooperation is higher, if agents are more risk averse, or income is riskier as measured by the standard deviation, while it does not depend on expected income.*

Proof. Replace the utility function (17), and $y^h = y$ and $y^l = y - p$ in (16), the equation determining $1/\delta^*$ for two possible income realizations. Leaving out the terms that are independent of the parameters of interest, we have

$$\frac{\left(-\frac{1}{A}\right) exp\left(-A\left(y-\frac{p}{2}\right)\right) - \left(-\frac{1}{A}\right) exp\left(-A\left(y-p\right)\right)}{\left(-\frac{1}{A}\right) exp\left(-Ay\right) - \left(-\frac{1}{A}\right) exp\left(-A\left(y-\frac{p}{2}\right)\right)},$$

which can be rewritten as

$$\frac{exp\left(Ap\right) - exp\left(\frac{1}{2}Ap\right)}{exp\left(\frac{1}{2}Ap\right) - 1} = exp\left(\frac{1}{2}Ap\right).$$

Thus $1/\delta^*$ only depends on A and p, and is independent of y, that is, of expected income. Further it is increasing in both A and p. □

Remark 1. *One may say that the correct measure of riskiness in the case of CARA preferences is the standard deviation, since, along with risk aversion, this is what determines cooperation in the informal insurance game.*

3.2 CRRA preferences

Suppose that both agents have standard isoelastic preferences, that is,

$$u\left(c\right) = \frac{c^{1-\sigma}}{1-\sigma}, \tag{18}$$

where $\sigma > 0$ is the coefficient of relative risk aversion, and $\sigma \neq 1$. For $\sigma = 1$, $u\left(c\right) = ln\left(c\right)$. We know that, in the case of CRRA preferences, if an agent, with given σ and wealth, is indifferent between accepting and not accepting a multiplicative risk, then this is true for any wealth level. In other words, what matters in the coefficient of variation of the gamble.

To fix ideas, suppose that $y^h = y$ and $y^l = (1-q)\,y$, with $y > 0$ and $0 < q < 1$. In this case, mean income is $\left(1 - \frac{q}{2}\right) y$, and the coefficient of variation is $\frac{2}{2-q}$. Notice that the coefficient of variation only depends on q, and is independent of y. Note that any mean - coefficient of variation combination can be reproduced by choosing q and y appropriately. As in the CARA case, we examine how $1/\delta^*$ depends on the parameters of the model.

Claim 2. *In the CRRA case, $1/\delta^*$ depends positively on σ and q, and is independent of y. That is, more cooperation is achieved, if agents are more risk averse, or income is riskier* *as measured by the coefficient of variation, while cooperation does not depend on expected income.*

Proof. In the appendix. □

Remark 2. *One may say that the correct measure of riskiness in the case of CRRA preferences is the coefficient of variation, since, along with risk aversion, this is what determines cooperation in the informal insurance game.*

Thus, $1/\delta^*$ is consistent with standard measures of risk aversion, and with measuring riskiness by the standard deviation (coefficient of variation), if preferences are of the CARA (CRRA) form. However, $1/\delta^*$ can be computed for any type of utility function, while it can still disentangle risk and expected value, and one can compare the riskiness of random variables with different means.

4 No aggregate uncertainty

Let us now turn to general, increasing and concave utility functions. This section looks at the case where, sharing risk perfectly, agents' consumption is completely smooth across states and over time. That is, agents only face idiosyncratic risk, aggregate income in the community is the same in all states of the world. For this, the two agents' incomes must be perfectly negatively correlated. Examining informal insurance in this case is related to a standard insurance setting, where a risk-averse agent can buy complete insurance from a principal, in other words, there is no background risk.

Since aggregate income is constant across states of the world and shared equally between the two agents, consumption of both agents is equal to per-capita income, $\overline{y}$. Then, (15) can be rewritten as

$$\begin{aligned}\frac{1}{\delta^*} &= 1 + \frac{u(\overline{y}) - \sum_s Pr(s)\, u(y_i(s))}{u(y^h) - u(\overline{y})} \\ &= 1 + \frac{u(\overline{y}) - u(CE^u)}{u(y^h) - u(\overline{y})}, \qquad (19)\end{aligned}$$

where CE^u denotes the certainty equivalent of the distribution Y when preferences are described by the function $u()$.

Note that the complete insurance case means putting strong restrictions on the possible income distributions. In particular, if some income $y_i(s)$ is earned with probability $Pr(s)$, then there must be another income realization $y_i(-s) = 2\overline{y} - y_i(s)$, where $2\overline{y}$ is the constant aggregate income, and it must occur with the same probability, that is, $Pr(-s) = Pr(s)$. In other words, the distribution must be symmetric.

This section conducts a number of comparative static exercises on how the level of cooperation, given by equation (19), depends on the characteristics of the utility function, and the income distribution, Y. In particular, I examine how $1/\delta^*$ depends on the concavity of the utility function, that is, on risk aversion. I also study how $1/\delta^*$ changes, if the riskiness of the income distribution changes in terms of a mean-preserving spread, and when ranking the riskiness of distributions is based on second-order stochastic dominance (SSD).

First, let us compare cooperation levels when risk aversion changes. A standard characterization states that agent j, with utility function $v()$, is more risk averse than agent i, with utility function $u()$, if and only if $v()$ is an increasing and concave transformation of $u()$. This is equivalent to saying that agent j's (Arrow-Pratt) coefficient of absolute risk aversion is uniformly greater than that of agent i. Denote by $\phi()$ the increasing and concave function that transforms $u()$ into $v()$, that is, $v() = \phi(u())$. Taking Y as given, denote by δ_v^* (δ_u^*) the discount factor above which perfect risk sharing is self-enforcing, if agents have utility function $v()$ ($u()$).

Proposition 2. *With no aggregate uncertainty, $1/\delta_v^* \geq 1/\delta_u^*$. That is, if agents are more risk averse in the sense of having a more concave utility function, then cooperation in the informal insurance game increases.*

Proof. Using the formula determining $1/\delta^*$ with no aggregate uncertainty, equation (19), $1/\delta_v^* \geq 1/\delta_u^*$ is equivalent to

$$\frac{v(\overline{y}) - v(CE^v)}{v(y^h) - v(\overline{y})} \geq \frac{u(\overline{y}) - u(CE^u)}{u(y^h) - u(\overline{y})}.$$

Replacing $\phi(u())$ for $v()$ yields

$$\frac{\phi(u(\overline{y}))-\phi(u(CE^v))}{\phi(u(y^h))-\phi(u(\overline{y}))} \geq \frac{u(\overline{y})-u(CE^u)}{u(y^h)-u(\overline{y})}. \tag{20}$$

Since $\phi()$ is increasing and concave, and $u\left(y^h\right) > u(\overline{y}) > u(CE^u) > u(CE^v)$, we know that

$$\frac{\phi(u(\overline{y}))-\phi(u(CE^v))}{u(\overline{y})-u(CE^u)} \geq \frac{\phi(u(\overline{y}))-\phi(u(CE^u))}{u(\overline{y})-u(CE^u)} \geq \frac{\phi\left(u\left(y^h\right)\right)-\phi(u(\overline{y}))}{u(y^h)-u(\overline{y})}.$$

Rearranging yields (20). □

Proposition 2 means that we have the desirable comparative static result between risk aversion and the level of cooperation, when complete insurance is achieved, using concavity of the utility function as the measure of risk aversion, and $1/\delta^*$ as the measure of cooperation. Proposition 2 is analogous to the well-known result that a more risk-averse agent is willing to pay more for formal, complete insurance, with the same measure of risk aversion.

In the case of formal insurance, we know that a decrease in wealth, or, equivalently, an increase in a lump-sum tax, makes risk-averse agents willing to pay more to avoid a given risk, if preferences exhibit nonincreasing absolute risk aversion (DARA). This comparative static result goes through to the informal insurance case as well, as the following corollary states.

Corollary 1. *If preferences are characterized by nonincreasing absolute risk aversion (DARA), then a decrease in wealth, or, an increase in a lump-sum tax, results in more cooperation.*

Proof. Follows from Proposition 2 and the well-known result that, under DARA, a decrease in wealth is equivalent to an increasing and concave transformation of the utility function. □

Let us now turn to riskiness. First, a mean-preserving spread on the income distribution is taken as the criterion for ranking the riskiness of random incomes. I examine how $1/\delta^*$ changes when riskiness according to this standard concept changes under either of the following two assumptions.

Assumption (a). Income may take maximum three values.

Assumption (b). The support of the income distribution is constant.

Under assumption (a) and no aggregate uncertainty, there are only three possible income states: hl (agent 1 earning high income y^h, and agent 2 getting y^l) and lh (the reverse), and both occur with probability Pr^{asym}[6], and in the third income state, both agents must earn $\overline{y}$.

To consider a mean-preserving spread in this case, let us define a new income distribution, $\widetilde{Y}$, as $\widetilde{y}^h = y^h + \epsilon$ and $\widetilde{y}^l = y^l - \epsilon$, with $\epsilon > 0$. Note that mean income does not change, that is, $\frac{\widetilde{y}^h + \widetilde{y}^l}{2} = \frac{y^h + y^l}{2} = \overline{y}$. In the third income state, nothing changes. Denote by $\widetilde{\delta}^*$ the corresponding discount factor above which perfect risk sharing is self-enforcing.

Under assumption (b), the extreme income realizations, y^h and y^l are kept constant, and the spread occurs on the "inside" of the distribution. Denote by $1/\widetilde{\delta}^*$, for this case as well, the level of informal insurance corresponding to the more risky income distribution, $\widetilde{Y}$.

Proposition 3. $1/\widetilde{\delta}^* \geq 1/\delta^*$, *that is, when there is no aggregate uncertainty and assumption (a) or (b) holds, if income is riskier in the sense of a mean-preserving spread, then cooperation is higher in the informal insurance game.*

Proof. Under assumption (a), (15) can be written as

$$\frac{1}{\delta^*} = 1 - Pr^{asym} + Pr^{asym} \frac{u\left(\overline{y}\right) - u\left(y^l\right)}{u\left(y^h\right) - u\left(\overline{y}\right)}. \tag{21}$$

Thus, in this case, $1/\widetilde{\delta}^* \geq 1/\delta^*$ is equivalent to

$$\frac{u\left(\overline{y}\right) - u\left(\widetilde{y}^l\right)}{u\left(\widetilde{y}^h\right) - u\left(\overline{y}\right)} \geq \frac{u\left(\overline{y}\right) - u\left(y^l\right)}{u\left(y^h\right) - u\left(\overline{y}\right)}.$$

Replacing for $\widetilde{y}^h$ and $\widetilde{y}^l$ gives

$$\frac{u\left(\overline{y}\right) - u\left(y^l - \epsilon\right)}{u\left(y^h + \epsilon\right) - u\left(\overline{y}\right)} \geq \frac{u\left(\overline{y}\right) - u\left(y^l\right)}{u\left(y^h\right) - u\left(\overline{y}\right)}. \tag{22}$$

Now, since $u()$ is increasing and concave, we know that

$$\frac{u\left(\overline{y}\right) - u\left(y^l - \epsilon\right)}{\overline{y} - y^l + \epsilon} \geq \frac{u\left(\overline{y}\right) - u\left(y^l\right)}{\overline{y} - y^l},$$

[6] If, for example, agent i earned y^h with probability $\pi > Pr^{asym}$, agent $-i$ (the other agent) would get y^h with a smaller probability $1 - \pi < Pr^{asym}$, the two agents' expected incomes would differ, thus they would not be ex-ante identical.

and

$$\frac{u\left(y^h+\epsilon\right)-u\left(\overline{y}\right)}{y^h+\epsilon-\overline{y}} \leq \frac{u\left(y^h\right)-u\left(\overline{y}\right)}{y^h-\overline{y}}.$$

Then, using the fact that $y^h - \overline{y} = \overline{y} - y^l$, dividing gives (22).

Under assumption (b), $1/\widetilde{\delta}^* \geq 1/\delta^*$ is equivalent to

$$\frac{u\left(\overline{y}\right)-u\left(\widetilde{CE}^u\right)}{u\left(y^h\right)-u\left(\overline{y}\right)} \geq \frac{u\left(\overline{y}\right)-u\left(CE^u\right)}{u\left(y^h\right)-u\left(\overline{y}\right)}, \tag{23}$$

where $\widetilde{CE}^u$ is the certainty equivalent of the riskier distribution $\widetilde{Y}$. It is well known that $\widetilde{CE}^u < CE^u$, thus (23) holds. □

Thus, in the complete insurance case, $1/\delta^*$ is consistent with a mean-preserving spread as the measure of riskiness, assumptions (a) or (b) being sufficient conditions. Now, let us consider second-order stochastic dominance (SSD) as the measure of riskiness. With a constant mean, the above result naturally extends to SSD, since an SSD deterioration is equivalent to a sequence of mean-preserving spreads. The result still holds if the dominated process has a lower mean, as the following corollary states.

Corollary 2. *In the complete insurance case, under assumption (b), if income is riskier in the sense of an SSD deterioration, then there is more informal insurance.*

Proof. Follows from the proof of Proposition 3, noting that, if $\widetilde{Y}$ is dominated by Y in the sense of SSD, then $\widetilde{CE}^u < CE^u$ for any $u()$ increasing and concave. □

Thus assumption (b) is a sufficient condition for the desirable comparative static result, using SSD to compare the riskiness of income distributions. Future work should determine necessary conditions.

5 With aggregate uncertainty

This section examines the case where agents must bear some consumption risk, even though they share risk perfectly. The community faces aggregate risk as well, while agents can only provide insurance to each other against idiosyncratic risks. In particular, I assume that income is realized independently for the two agents. Remember that to have no aggregate uncertainty, as in section 4, income realizations have to be perfectly negatively correlated across agents. As in the standard insurance setting when the agent cannot buy complete insurance, one may also say that there is background risk. I am interested in what goes through from the results of section 4.

Let us consider risk aversion first. Remember that $u()$ and $v()$ are two utility functions, and we have assumed that an agent with utility function $v()$ is more risk averse than an agent with utility function $u()$. Remember also that δ_v^* (δ_u^*) denotes the discount factor above which perfect risk sharing is self-enforcing, if agents have utility function $v()$ ($u()$). The following assumption is sufficient to guarantee that the desirable comparative static result holds.

Assumption (c). $u(y_1(s)) + u(y_2(s)) \leq 2u(\overline{y})$, where $\overline{y} = \frac{y^h + y^l}{2}$, for all s where $y_1(s) \neq y_2(s)$.

This assumption means that there is no asymmetric state where the expected utility in autarky would be higher than the utility from consuming $\overline{y}$.

Proposition 4. *With aggregate uncertainty, under assumption (c), $1/\delta_v^* \geq 1/\delta_u^*$. That is, if agents are more risk averse in the sense of having a more concave utility function, then cooperation is higher in the informal insurance game.*

Proof. Using the formula determining $1/\delta^*$, equation (15), for $1/\delta_v^* \geq 1/\delta_u^*$ to hold it is sufficient that

$$\frac{v\left(\frac{y_1(s)+y_1(-s)}{2}\right) - \frac{1}{2}\left[v(y_1(s)) + v(y_1(-s))\right]}{v(y^h) - v(\overline{y})} \geq \frac{u\left(\frac{y_1(s)+y_1(-s)}{2}\right) - \frac{1}{2}\left[u(y_1(s)) + u(y_1(-s))\right]}{u(y^h) - u(\overline{y})}, \forall s. \quad (24)$$

Denote by $Ey(s)$ mean income at state s, and by $CE^u(s)$ the certainty equivalent at state s, when preferences are described by the utility function $u()$, that is, $u(CE^u(s)) = \frac{1}{2}u(y_1(s)) + \frac{1}{2}u(y_1(-s))$. Then, (24) can be written as

$$\frac{v(Ey(s)) - v(CE^v(s))}{v(y^h) - v(\overline{y})} \geq \frac{u(Ey(s)) - u(CE^u(s))}{u(y^h) - u(\overline{y})}.$$

To complete the proof, one may use the same argument as in the proof of Proposition 2. □

Here I put a restriction on the income process that is a sufficient condition for more risk aversion to increase voluntary insurance. One could also follow another approach, like Ross (1981) in the case of formal insurance, to find a stronger measure of risk aversion.

Let us now turn to riskiness, in particular, how $1/\delta^*$ changes if there is a mean-preserving spread on the income distribution. I provide counterexamples to the expected comparative static result. It turns out to be sufficient to examine the simplest possible income distribution.

Suppose that income may only take two values, high or low, denoted y^h and y^l, respectively, as before. Let π denote the probability of earning y^h. Then $Pr^{asym} = \pi(1-\pi)$. Now, let us define a new, more risky income distribution, $\widehat{Y}$, in the sense of a mean-preserving spread. Let the new high income realization be $\widehat{y}^h = y^h + \epsilon$, with $\epsilon > 0$. To keep mean income constant, $\widehat{y}^l$ must equal $y^l - \frac{\pi}{1-\pi}\epsilon$, with $\epsilon < \frac{1-\pi}{\pi}y^l$. Note that in this case consumption in the asymmetric states is $\frac{\widehat{y}^h+\widehat{y}^l}{2} = \frac{y^h+y^l}{2} + \frac{1-2\pi}{1-\pi}\frac{\epsilon}{2}$. Denote the corresponding level of cooperation by $1/\widehat{\delta}^*$.

Proposition 5. *It is not true in general that $1/\widehat{\delta}^* \geq 1/\delta^*$. That is, with aggregate uncertainty, a mean-preserving spread on incomes may result in less cooperation, even when income may take only two values.*

Proof. Let us construct a counterexample. Take $y^h = 1.5$, $y^l = 0.55$, $\pi = 0.6$ (so mean income is $0.6 \cdot 1.5 + 0.4 \cdot 0.55 = 1.12$), thus $Pr^{asym} = \pi(1-\pi) = 0.6 \cdot 0.4 = 0.24$, and $\epsilon = 0.2$. It follows that $\frac{y^h+y^l}{2} = 1.025$, and $\widehat{y}^h = 1.7$, $\widehat{y}^l = 0.25$, and $\frac{\widehat{y}^h+\widehat{y}^l}{2} = 0.975$. The mean is now $0.6 \cdot 1.7 + 0.4 \cdot 0.25 = 1.12$. Thus the distribution $\widehat{Y}$ is indeed a mean-preserving spread of Y. Consider the utility function

$$u(c) = \begin{cases} c^{0.8} & \text{if } c < 1 \\ c^{0.1} & \text{if } c > 1 \end{cases},$$

and smooth it appropriately in a small neighborhood of 1. This utility function could represent the preferences of a loss-averse agent. Replacing the above values in (16), we have

$$\frac{1}{\delta^*} = 1 - 0.24 + 0.24\frac{1.025^{0.1} - 0.55^{0.8}}{1.5^{0.1} - 1.025^{0.1}} = 3.12,$$

and

$$\frac{1}{\widehat{\delta}^*} = 1 - 0.24 + 0.24\frac{0.975^{0.8} - 0.25^{0.8}}{1.7^{0.1} - 0.975^{0.8}} = 2.85,$$

which contradicts $1/\widehat{\delta}^* \geq 1/\delta^*$.

The result does not hinge on the fact that $\pi > \frac{1}{2}$, and that therefore consumption in the asymmetric states, hl and lh, decreases. Take $y^h = 1.5$, $y^l = 0.495$, $\pi = 0.1$, and $\epsilon = 0.6$, and consider the same utility function as above. This specification provides another counterexample, since $1/\delta^* = 1.84$ and $1/\widehat{\delta}^* = 1.35$. $\square$

The intuition behind this result is the following. In the case of incomplete insurance, when income becomes riskier in the sense of a mean-preserving spread, not only the spread between the high and low income realizations changes, but also consumption in the asymmetric states. As a result, the transfers $\frac{y^h+y^l}{2} - y^l = y^h - \frac{y^h+y^l}{2}$ are not just increased to $\frac{y^h+y^l}{2} - y^l + \frac{1}{1-\pi}\frac{\epsilon}{2} = y^h - \frac{y^h+y^l}{2} + \frac{1}{1-\pi}\frac{\epsilon}{2}$, but they also occur at consumption levels that are shifted by $\frac{1-2\pi}{1-\pi}\frac{\epsilon}{2}$ at the mean. Because of this shift, the utility gain of insurance represented by $u\left(\frac{y^h+y^l}{2}\right) - u\left(y^l\right)$, and the loss of insurance represented by $u\left(\frac{y^h+y^l}{2}\right) - u\left(y^h\right)$ are evaluated at a different consumption level for the income distribution Y than for $\widehat{Y}$. The curvature of the utility function may differ sufficiently at the two consumption levels, so that the ratio between the utility gain and loss of informal insurance changes in an ambiguous way, when a mean-preserving spread occurs on the income distribution. In particular, the level of cooperation may decrease.

This result points out that, when agents share risk informally, determining how much consumption variability they have to deal with is a rather complex issue, since the link between income risk, in some standard sense, and consumption risk is not straightforward. This is the consequence of the interplay of idiosyncratic and aggregate risk. See also Attanasio and Ríos-Rull (2000), who show that aggregate insurance may reduce welfare, when agents share (idiosyncratic) risk informally.

How to reconcile this negative result? Aggregate risk should be kept constant, while idiosyncratic risk increases. To do this, some negative correlation between the income realizations of the two agents has to be reintroduced. This can indeed work, as the following example demonstrates.

Example. Let us reconsider the first example of the proof above. The original income distribution, Y, was $y^h = 1.5$, $y^l = 0.55$, with the probability of the high income realization $\pi = 0.6$, that is, $Pr^{asym} = 0.24$. The second, more risky income distribution, $\widehat{Y}$, was $\widehat{y}^h = 1.7$, $\widehat{y}^l = 0.25$, with $\pi = 0.6$ still. Expected individual income is 1.12 for both agents, while expected aggregate income is 2.24 for both income distributions. We wanted to increase idiosyncratic risk, however, aggregate risk has also increased. In particular, the standard deviation of the distribution of aggregate income has increased from 0.5472 to 0.8352.[7] Now, let us introduce some negative correlation between the income realizations of the two agents for $\widehat{Y}$, to match the standard deviation of Y. This can be achieved my setting $Pr^{asym} = 0.364$, and decreasing the probability of the hh and ll states by 0.124 each. Let us denote the level of informal insurance by $1/\check{\delta}^*$ in this case. Then $1/\delta^* = 3.12$ as before, but $1/\check{\delta}^* = 3.81$, thus, keeping aggregate risk constant, cooperation in the informal insurance game increases as a result of a mean-preserving spread on the income distribution.

6 Discussion on measuring informal insurance

Risk theorists have devoted a lot of attention to formal insurance contracts, that occur between a risk-averse agent and an insurance company. The first issue is to measure the level of insurance, that is, “how much?” insurance occurs. In the case of formal insurance, the answer is simple: we can measure insurance in money units. The second issue is to relate insurance to risk preferences and the riskiness of a random variable, or gamble. With appropriate measures of risk aversion and riskiness, we would like to have comparative static results like “if the agent is more risk averse, she is willing to pay more to avoid a given gamble”, and “a risk-averse agent is willing to pay more to avoid a riskier gamble”. See Pratt (1964), Arrow (1965), Hadar and Russell (1969), Rothschild and Stiglitz (1970), Ross (1981), Jewitt (1987, 1989), and others, and Gollier (2001) for a summary.

[7]Note that speaking about the standard deviation or the coefficient of variation is equivalent here, since the mean doesn’t change.

This paper, considering an informal insurance game, addresses similar issues. In particular, it examines how risk preferences and riskiness of agents' income together determine cooperation in the case of voluntary insurance, and aims to establish the type of comparative static results that exist for the case of formal insurance. In this section I look at how δ^*, the discount factor above which perfect risk sharing is self enforcing, is related to the complicated object, the set of state dependent intervals, that is the solution of the informal insurance game for any discount factor. I also examine how it relates to the insurance transfers and the smoothness of consumption across income states.

To do this, let us reconsider the numerical example presented in Ligon, Thomas, and Worrall (2002). Suppose that there are two agents with isoelastic utility and with a coefficient of relative risk aversion equal to 1, that is, $u() = ln()$. Income is independently and identically distributed (i.i.d.) across agents and time, and it may take two values, high ($y^h = 20$, say) or low (y^l=10)[8]. The probability of the low income realization is 0.1. Remember that when $\delta = \delta^*$, or whenever perfect risk sharing occurs and Pareto weights are equal, aggregate income should always be shared equally. This means that in the asymmetric states a transfer of 5 should be made, thus both agents consume 15.

Let us also consider an alternative scenario where the income distribution is as before, but agents are more risk averse. Denote the new utility function by $v()$. Let the coefficient of relative risk aversion be constant and equal to 1.5, thus $v(c) = c^{1-\sigma}/(1-\sigma) = c^{-0.5}/-0.5$.

The aim of this exercise is to compare the solution of the risk sharing with limited commitment model in these two cases. In particular, we first look at the optimal state-dependent intervals on the ratio of marginal utilities, that fully characterize the solution, as a function of the discount factor. I consider discount factors between 0.84 and 0.99. Then I examine what δ^*, tells us about the solution, and how it is related to the insurance transfers and the consumption allocation. The computations have been done using the software R (see www.r-project.org).

The black lines in Figure 1 reproduce figure 1 in Ligon, Thomas, and Worrall (2002), that

[8]The graph is the same for any y^h and y^l, if $y^l = 0.5y^h$ holds. The transfers and consumptions will be different, of course.

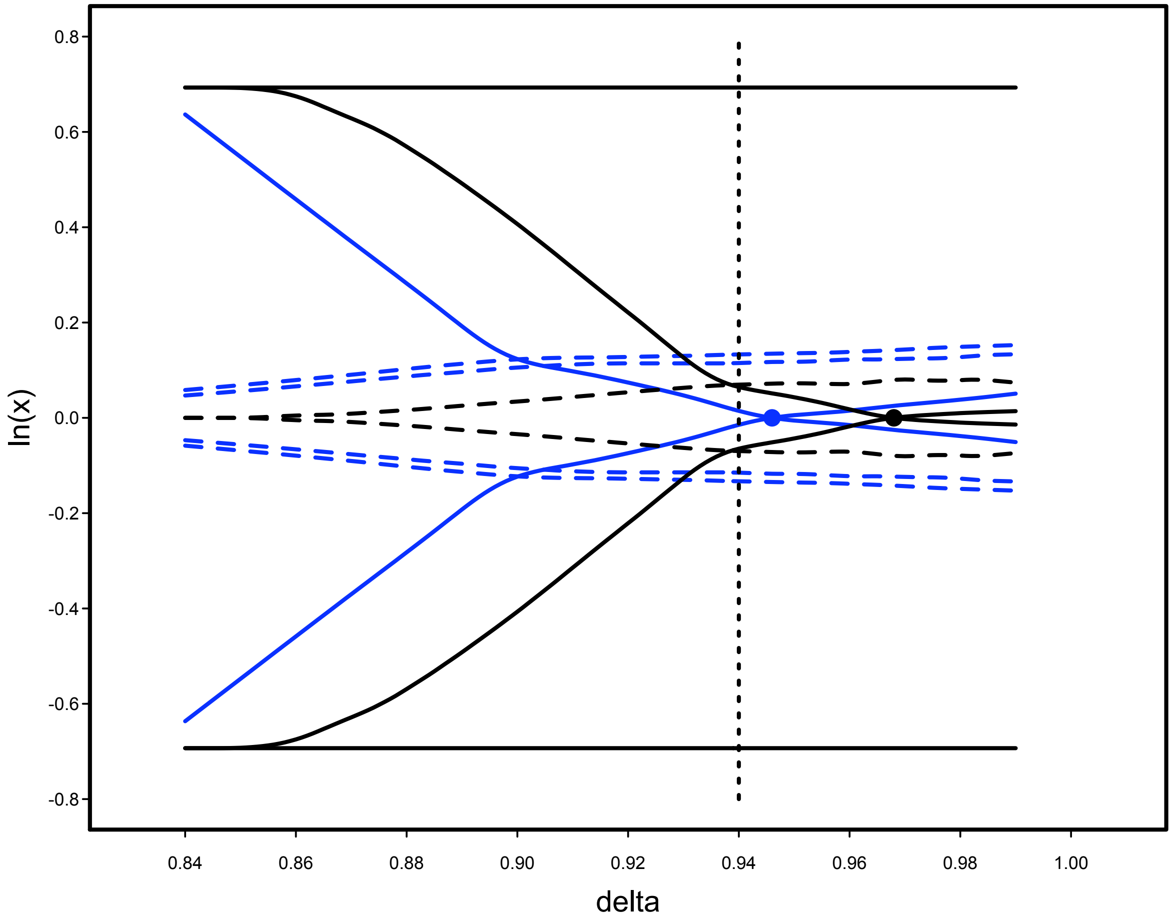

Figure 1: The optimal intervals of $ln\,(x)$ as a function of δ. The black lines show the optimal intervals on the (logarithm of the) ratio marginal utilities for the utility function $ln(c)$ (as in Ligon, Thomas, and Worrall (2002)), and the blue lines for $c^{1-\sigma}/(1-\sigma)$ with $\sigma = 1.5$. The dots represent δ^*. See more details in the main text.

represents the logarithm of x, the ratio of marginal utilities, as a function of the discount factor, δ. The dashed lines represent the optimal intervals for the symmetric states, hh and ll (the two coincide with logarithmic utility), while the solid lines are the intervals for the asymmetric states. The blue lines in Figure 1 show the corresponding intervals when $\sigma = 1.5$, that is, when agents are more risk averse.

First of all, let us look at the case where $\delta = 0.94$. For this discount factor, all the intervals overlap, except for the ones for states hl and lh (see the intervals along the vertical, dotted line in Figure 1). Then, the ratio of marginal utilities, after a sufficient number of periods, will only take two values, $\overline{x}^{hl}$ and $\underline{x}^{lh} = 1/\overline{x}^{hl}$. For the utility function $u()$, these numbers are 0.940 any 1.064. When agents' preferences are described by the more concave function $v()$, they equal 0.990 and 1.010. It follows from the first order conditions that the insurance transfers in the asymmetric states, hl and lh, are 4.53 and 4.92, for the utility functions $u()$ and $v()$, respectively. This also means that, if agents are more risk averse, consumption is smoother across states, so agents achieve more insurance. Note also that we are very close to the first-best transfer, 5, in both cases.

Now, notice that for any discount factor, the blue intervals, that belong to the case when agents are more risk averse, are wider. This means that a wider range of x's are possible with voluntary participation, in other words, agents cooperate more. Remember that, in the case where perfect risk sharing is self-enforcing, all the intervals overlap. On the other hand, when no informal insurance is possible, that is, when agents stay in autarky, each interval is just one point. Thus, if the intervals are wider, we may say that there is more insurance.

Finally, in Figure 1, the dots represent the discount factor above which perfect risk sharing is self-enforcing, δ^*. The black dot represents $\delta^* = 0.964$ for the utility function $u()$, while the blue dot is $\delta^* = 0.943$ that belongs to the more concave utility function $v()$. Notice that, as the dot moves to the left, the optimal intervals also move to the left, thus they become wider. Thus, one may capture the changes in the intervals, and thereby the changes in the transfers and the consumption allocation, by the scalar δ^*. Future research should determine how well δ^* may characterize the solution in more complicated settings.

7 Conclusion

This paper has shown a way to characterize cooperation in a widely-used informal insurance game, and made a first attempt to relate it to riskiness and risk aversion. In particular, I defined the level of cooperation, denoted $1/\delta^*$, as the reciprocal of the discount factor above which perfect risk sharing is self-enforcing. Comparative static results include that, if the utility function is more concave, that is, agents are more risk averse, $1/\delta^*$ is higher. However, in the case with aggregate uncertainty, a mean-preserving spread on the income process may decrease cooperation. This is because of the interplay of idiosyncratic and aggregate risk.

This paper has also shown that, in a simple setting, the comparative static results relating to the concavity of the utility function and the riskiness of the distribution of income go through to insurance transfers and the smoothness of consumption.

Let me conclude with a remark on measuring risk. Consider two simple distributions, Y and Z, that both have two possible realizations, high or low, determined by the toss of a fair coin. Y yields 1 or 2 euros, while Z gives 3 or 100. SSD or the recent measure of riskiness proposed by Aumann and Serrano (2008) tell us that Y is more risky, since it yields a lower payoff in all states of the world. However, Z seems to involve more variation. The standard deviation or the coefficient of variation would tell us that Z is indeed more risky, but these are right measures only for the CARA and CRRA cases, respectively. Supposing either preferences, $1/\delta^*$ gives a ranking that is consistent with the right measure of riskiness, the standard deviation or the coefficient of variation. It may also say something about the risk agents actually want and can insurance against in the case of more general preferences.

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8 Appendix

Proof of Claim 2:

Replace the utility function (18) and $y^h = y$ and $y^l = (1-q)\,y$ in the equation determining $1/\delta^*$ in the case of two possible income realizations, equation (21). For $\sigma \neq 1$, this gives

$$\frac{1}{\delta^*} = \frac{1}{2} + \frac{1}{2}\frac{\frac{\left(\left(1-\frac{q}{2}\right)y\right)^{1/\sigma}}{1-\sigma} - \frac{((1-q)y)^{1/\sigma}}{1-\sigma}}{\frac{y^{1-\sigma}}{1-\sigma} - \frac{\left(\left(1-\frac{q}{2}\right)y\right)^{1/\sigma}}{1-\sigma}},$$

which can be rewritten as

$$\frac{1}{\delta^*} = \frac{1}{2}\frac{(1-q)^{1-\sigma}-1}{\left(1-\frac{q}{2}\right)^{1-\sigma}-1}. \tag{25}$$

For $\sigma = 1$, we have $1/\delta^* = ln\,(1-q)\,/\left(2ln\left(1-\frac{q}{2}\right)\right)$. Thus $1/\delta^*$ only depends on σ and q, and is independent of y, that is, of mean income.

Now, I want to show that $\frac{\partial 1/\delta^*(\sigma,q)}{\partial\sigma} > 0$ and $\frac{\partial 1/\delta^*(\sigma,q)}{\partial q} > 0$. Let us suppose that $\sigma \neq 1$. The results generalize to $\sigma = 1$ taking limits. Let us differentiate equation (25) with respect to σ first. This gives

$$\begin{aligned} sign\left(\frac{\partial 1/\delta^*(\sigma,q)}{\partial\sigma}\right) &= sign\left(\frac{1}{2}\left[(1-q)^{1-\sigma}\,ln\,(1-q)\,(-1)\left(\left(1-\frac{q}{2}\right)^{1-\sigma}-1\right)\right.\right.\\ &\left.\left.-\left(1-\frac{q}{2}\right)^{1-\sigma} ln\left(1-\frac{q}{2}\right)(-1)\left((1-q)^{1-\sigma}-1\right)\right]/\left(\left(1-\frac{q}{2}\right)^{1-\sigma}-1\right)^2\right)\\ &= sign\left(ln\left(1-\frac{q}{2}\right)\left(1-(1-q)^{\sigma-1}\right) - ln\,(1-q)\left(1-\left(1-\frac{q}{2}\right)^{\sigma-1}\right)\right)\\ &= sign\left(\frac{1-(1-q)^{\sigma-1}}{ln\,(1-q)} - \frac{1-\left(1-\frac{q}{2}\right)^{\sigma-1}}{ln\left(1-\frac{q}{2}\right)}\right), \end{aligned}$$

where the third line follows after dividing by $(1-q)^{1-\sigma}\left(1-\frac{q}{2}\right)^{1-\sigma} > 0$, and the last line follows dividing by $ln\,(1-q)\,ln\left(1-\frac{q}{2}\right) > 0$. We know that $0 < \frac{q}{2} < q < 1$, thus $0 < 1-q < 1-\frac{q}{2} < 1$. What remains to be shown is that the function

$$f(z) \equiv \frac{1-z^{\sigma-1}}{ln(z)}$$

is decreasing in z, $z \in (0,1)$. To do this, let us differentiate $f(z)$ with respect to z. This gives

$$sign\left(\frac{\partial f(z)}{\partial z}\right) = sign\left(\frac{-(\sigma-1)z^{\sigma-2}ln(z) - \frac{1}{z}(1-z^{\sigma-1})}{(ln(z))^2}\right)$$
$$= sign\left((1-(\sigma-1)ln(z))z^{\sigma-1}-1\right).$$

Note that $\lim_{z\to 1}(1-(\sigma-1)ln(z))z^{\sigma-1}-1=0$. Now, to show that $(1-(\sigma-1)ln(z))z^{\sigma-1}-1<0$, we only have to establish that $g(z)\equiv(1-(\sigma-1)ln(z))z^{\sigma-1}-1$ is increasing in z for $z\in(0,1)$. Taking derivatives with respect to z gives

$$sign\left(\frac{\partial g(z)}{\partial z}\right) = sign\left((\sigma-1)z^{\sigma-2}-(\sigma-1)\left(\frac{1}{z}z^{\sigma-1}+ln(z)(\sigma-1)z^{\sigma-2}\right)\right)$$
$$= sign\left(-(\sigma-1)^2 ln(z) z^{\sigma-2}\right).$$

The first term is positive, the second is negative, the third is positive, and all this is multiplied by (-1), thus $\frac{\partial g(z)}{\partial z}$ is positive. It follows that $\frac{\partial f(z)}{\partial z}$ is negative, and that $\frac{\partial 1/\delta^*(\sigma,q)}{\partial\sigma}$ is positive.

Now, let us differentiate equation (25) with respect to q. This gives

$$sign(\frac{\partial 1/\delta^*(\sigma,q)}{\partial q}) = sign\left(\frac{1}{2}(1-\sigma)\left[(1-q)^{-\sigma}(-1)\left(\left(1-\frac{q}{2}\right)^{1-\sigma}-1\right)-\right.\right.$$
$$\left.\left.-\left((1-q)^{1-\sigma}-1\right)\left(1-\frac{q}{2}\right)^{-\sigma}\left(-\frac{1}{2}\right)\right]/\left(\left(1-\frac{q}{2}\right)^{1-\sigma}-1\right)^2\right)$$
$$= sign\left((1-\sigma)\left[\frac{1}{2}\left((1-q)^{1-\sigma}-1\right)\left(1-\frac{q}{2}\right)^{-\sigma}-\right.\right.$$
$$\left.\left.-(1-q)^{-\sigma}\left(\left(1-\frac{q}{2}\right)^{1-\sigma}-1\right)\right]\right)$$
$$= sign\left((1-\sigma)\left[\left(1-\frac{q}{2}\right)^{\sigma}-\frac{1}{2}\left((1-q)^{\sigma}+1\right)\right]\right).$$

The last line follows after dividing by $(1-q)^{-\sigma}>0$ and $\left(1-\frac{q}{2}\right)^{-\sigma}>0$. We have to consider two cases.

- $\sigma<1$. Now $1-\sigma>0$, so we have to show that $\left(1-\frac{q}{2}\right)^{\sigma}-\frac{1}{2}\left((1-q)^{\sigma}+1\right)>0$.
- $\sigma>1$. In this case $1-\sigma<0$, so we have to show that $\left(1-\frac{q}{2}\right)^{\sigma}-\frac{1}{2}\left((1-q)^{\sigma}+1\right)<0$.

We may rewrite this last expression as

$$\left(1-\frac{q}{2}\right)^{\sigma}-\frac{1^{\sigma}+(1-q)^{\sigma}}{2}. \tag{26}$$

Note that $1 - \frac{q}{2}$ is the mean of 1 and $1 - q$. Let us define $h(z) \equiv z^{\sigma}$. So what we are comparing is the mean (a convex combination) of the values $h(1)$ and $h(1-q)$ to the value at the mean, that is, $h\left(\frac{1+1-q}{2}\right) = h\left(1 - \frac{q}{2}\right)$.

- $\sigma < 1$. $h(z)$ is concave, thus $h\left(1 - \frac{q}{2}\right) > \frac{h(1)+h(1-q)}{2}$. It follows that (26) is positive, as I wanted to show.

- $\sigma > 1$. $h(z)$ is convex, thus $h\left(1 - \frac{q}{2}\right) < \frac{h(1)+h(1-q)}{2}$. It follows that (26) is negative, and this is what I wanted to show.