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Socially Optimal Child-Related Transfers and Personal Income Tax with Endogenous Fertility

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Socially Optimal Child-Related Transfers and Personal Income Tax with Endogenous Fertility

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Abstract

To compare the systems of child benefits and of family tax deductions, we create a model with endogenous fertility and basic income, also financed from proportional wage taxes. Pensioners are neglected but younger and older workers are distinguished: the former raise children and receive child benefits, while the latter not. Through the balance equation, current average fertility depends on past average fertility. To have a socially optimal positive child benefit, past average fertility has to be less than 1. The deduction's efficiency is presumably lower than the benefit's and may even be lower than that of pure basic income.

Keywords: progressive income tax, child benefits, family tax deductions, endogenous fertility

JEL classification: J13

Optimális gyermekszámfüggő transzferek és személyi jövedelemadó endogén termékenység esetén

Simonovits András

Összefoglaló

A családi pótlék és a családi adókedvezmény összehasonlításához egy olyan modellt konstruálunk, amelyben a termékenység endogén, és az alapjövedelmet arányos személyi jövedelemadóból fedezik. A nyugdíjasokat elhanyagoljuk, de a fiatal és az idős dolgozókat megkülönböztetjük: az előzők gyermeket nevelnek és családi pótlékot kapnak, az utóbbiak nem. Egy ilyen modellben a jelenlegi átlagos termékenység az előző átlagos termékenységtől függ, dinamikussá téve a modellt. Legreálisabb paraméteregyütteseinkre a kedvezmény hatékonysága kisebb, mint a pótléké, és elmaradhat még a tiszta alapjövedelem hatékonyságától is.

Tárgyszavak: progresszív jövedelemadó, családi pótlék, családi adókedvezmény, endogén termékenység

JEL kód: J13

traffer5, ep/2

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July 24, 2015

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Abstract*

To compare the systems of child benefits and of family tax deductions, we create a model with endogenous fertility and basic income, also financed from proportional wage taxes. Pensioners are neglected but younger and older workers are distinguished: the former raise children and receive child benefits, while the latter not. Through the balance equation, current average fertility depends on past average fertility. To have a socially optimal positive child benefit, past average fertility has to be less than 1. The deduction's efficiency is presumably lower than the benefit's and may even be lower than that of pure basic income.

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1. Introduction

After the baby boom has petered out, below-reproduction fertility rates have become a great problem in a number of developed countries. As a reaction, the concerned governments have expanded fertility-related transfer schemes to support families and promote fertility. Note that these schemes vary across time and space. In addition to free school and health care, various financial family support systems exist. To name just the two main types: the *child benefit* is increasing with the number of children (below working age), while the *family tax deduction* is a child benefit which only applies up to the income tax obligation. These fertility-related systems operate together with an income-dependent tax system and even interact with the public pension system. In this paper, we will create and analyze a model of optimal income- and fertility-related transfers when fertility is endogenous. Our main message is as follows: when the past average fertility is below 1, the partial replacement of the personal income tax by the child benefit system increases social welfare; its replacement with family tax deduction may diminish welfare.

To model such transfer systems, the framework of a Stackelberg-game is used: the government announces a transfer rule, and calculating with the transfers, the workers decide on their individual optima. Anticipating these reactions, the government determines the transfer rule by maximizing a social welfare function. (Even if the government does not maximize any social welfare function, this technique is appropriate for evaluating various transfer systems.) Taking into account individual and social budget constraints calls for a general equilibrium analysis.

In models of endogenous fertility (e.g. Becker, 1960; 1991; Becker and Barro, 1988), when workers decide on their fertility, they consider that more children means that younger workers consume less but have more joy. As is usual, a unisex world is heroically assumed, where the number of children can be any positive real, including the irrational number $\sqrt{2}$.

The efficiency of a public pension system is proportional to the fertility, therefore family support and pension appear as Siamese twins (e.g. Groezen, Leers and Meijdam (2003), for short, GLM theoretically and Gábos, Gál and Kézdi (2009) empirically). Nonetheless, in this paper we shall neglect the latter and try to deepen the analysis of the former. To study the properties of a child benefit system, one can neglect income (wage) taxation. But to analyze a family tax deduction system, one needs also model a tax system which redistributes incomes from richer adults to poorer ones, namely provides basic income to every adult. And then it is natural to model wage heterogeneity as well. Unfortunately, this dimension has been much neglected in the *theoretical* literature on fertility-related transfers. Making up this omission is the main aim of the present paper. In contrast, certain *applied* modelers analyzed these complications in much detail but they constructed the social welfare function to fit the real data rather than deriving the optimum (e.g. Haan and Wrohlich, 2011).

Earlier models (e.g. GLM and Simonovits (2013), (2014)) took the per child raising costs independent of after-tax family income: π, while the average wage is unity. Following Simonovits (2015a), we shall rather assume that the raising costs are proportional to post-tax incomes: $k = (z + \varphi n)\pi$, where z is the old-age worker's net income, φ is the benefit rate and n is the fertility. For a low enough child benefit rate, low wage earners' after tax income is much lower than 1. This modification makes the individual budget condition nonlinear in fertility, hence more complicated than its linear precursors. Appendix B discusses the qualitative and quantitative impact of the approximation.

We even modified the framework of Simonovits (2015a): to the generation of young workers who raise their children, we added the generation of old workers not raising children. (Note that Pestieau and Ponthier (2015) also have two rather than one working generation but they allow both young and old workers raise children.) That way we have made the working period twice longer than the raising period. On the other hand, the originally static model has become dynamic: the current fertility strongly depends on past average fertility, causing complications.

To simplify the analysis, first a child benefit system, then a family tax deduction system is studied. In our model, the personal income tax is simply proportional to the wage. Direct redistribution among the adults is achieved via a uniform basic income, while the child benefit is proportional to the number of children. For simplicity, the young worker's utility function is an additive logarithmic function of the young- and the old-age consumption and of fertility (cf. GLM).

Our social welfare function is the expected value of the young workers' maximal lifetime utilities. (Unlike in previous models of ours, to make the optimal child benefit positive, now there is no need to assume that the government attaches a higher preference for having children than the parents do.) Since the (pre-tax) wages, the labor supply and the share of reported earning are given, we also fix the tax rate, and look for the socially optimal basic income plus either the benefit rate or the tax deduction rate. (Otherwise we could claim the social optimality of total income redistribution. Furthermore, for a fixed tax rate, only the introduction of a basic income makes room for a changing child benefit rate.)

We have three qualitative analytical results for the child benefit system. Corollary and Theorem 1: the wage-specific unbalanced fertility rate is an increasing function of the child benefit rate, a decreasing function of the basic income and again an increasing function of the *balanced* child benefit rate (when the exogenously given tax covers both the benefits and the basic income). Theorem 2: the steady-state fertility is an increasing function of the balanced child benefit rate. Theorem 3: for any past average fertility rate below 1, a low enough benefit is better than no benefit.

Turning to the family tax deduction system and confining the examination to two types (with low and high wages), the optimal tax deduction rate is a corner solution, where the tax obligation of the low-paid worker just equals the tax deduction. On the one hand, if the tax rate is high, the wage distribution is modestly unequal and the past average fertility is high, then the benefit system is more efficient than the deduction system is. Then even the pure basic income system may overtake the deduction system. On the other hand, if the tax rate is low, the wage distribution is very unequal and the past average fertility is low, then the opposite holds. It is to be underlined that in practice, the family tax deduction system may be very far from this optimum, because the personal income tax rate is so low that only the high earners can use the deduction sufficiently. Though the model is rather theoretical, we have to support our incomplete analytical results with numerical illustrations.

We call the reader's attention to our model's limitations within the theoretical field: sterility is neglected (for asymmetric information, see Cremer, Gahvari and Pestieau, 2008), the labor supply is fixed (for flexible labor, see Fenge and Meier, 2009) and differences in the relative raising costs and the utility functions, especially in the parameter of relative utility of a child are glossed over (cf. Simonovits, 2013 and 2014). The role of social norms are also neglected (for its analytical treatment, see e.g. Lindbeck, Nyberg and Weibull (1999). For a rich empirical discussion of these issues in developing countries, see Banerjee and Duflo (2011), Chapter 5). The activities occurring outside the market are especially important in raising children (e.g. Lee and Mason eds. (2011) and Gál, Szabó and Vargha, 2014) but they are also skipped over. Murphy (2006) applied *agent-based modeling* to investigate the role of assortative mating on population growth. Further research will clarify how much the message of the paper changes if these complexities are taken into account.

The remainder is organized as follows: Section 2 investigates the minimal model of child benefits. Section 3 discusses a similar model of family tax deduction. Section 4 numerically checks the robustness of our results and Section 5 concludes. Proofs are relegated to Appendix A and an accepted approximation is analyzed in Appendix B.

2. Child benefit system

The numbers appearing in the paper are generally positive real ones. In this section, first we shall determine the individually optimal fertility under child benefits, then deduce the macro relations and finally illustrate the results numerically.

Individually optimal fertility

In our model, there is a unisex population, where young workers give birth to children, raise them, together with old workers they pay taxes to finance child benefits and a universal basic income. For simplicity, there is neither saving nor discounting.

Every young worker in our population is characterized by her pre-tax wage w which is preserved for old age. The wages are distributed according to a probability distribution function F and the distribution defines a corresponding expectation operator $\mathbf{E}$. Any young worker can freely choose the number of her children, denoted by n. The transfer system has two objectives: (i) to diminish pre-tax wage inequalities and (ii) to finance a part of the raising costs. Every worker receives a basic income γ. A worker, earning wage w pays tax θw, $0 \le \theta \le 1$. A young worker, who has n children, also receives child benefits φn, $\varphi > 0$ being the child benefit rate.

Denoting the net-of-tax rate by $\hat{\theta} = 1 - \theta$, the old worker's net income is equal to

$$z = \hat{\theta} w + \gamma \tag{1a}$$

and the young worker's net income is

$$z + \varphi n. \tag{1b}$$

We assume that the raising cost is proportional to the number of children and the net income. Denoting the proportionality constant by $\pi > 0$, the raising cost is equal to $\pi n(z + \varphi n)$, therefore the old and young worker's consumption are respectively given by

$$d = z \tag{2a}$$

and

$$c = (1 - \pi n)(z + \varphi n), \quad \text{where} \quad 0 < n < \frac{1}{\pi}. \tag{2b}$$

To avoid absurd cases, we assume that the child benefit is always nonnegative and not greater than the narrow raising cost: $0 \le \varphi \le \pi z_{\rm m}$, where $z_{\rm m} = \hat{\theta} w_{\rm m} + \gamma$.

Assume that any young worker chooses her fertility to maximize an additive logarithmic utility function

$$U(c, n, d) = \log c + \zeta \log n + \log d, \tag{3}$$

where $\zeta > 0$ is the relative individual utility of having children with respect to that of adult consumption.

Inserting (2) into (3) yields the reduced utility function

$$u(n) = \log(z + \varphi n) + \log(1 - \pi n) + \zeta \log n + \log z. \tag{4}$$

We assume that the workers neglect the impact of their decisions on the tax balance described in (9) below.

Equating $u's$ derivative to zero provides the young worker's necessary optimality condition:

$$0 = u'(n) = \frac{\varphi}{z + \varphi n} - \frac{\pi}{1 - \pi n} + \frac{\zeta}{n}. \tag{5}$$

Since $u'(n)$ is a decreasing function and $u(0) = -\infty$, the root is a global maximum.

At this point we shall consider the simplest case when there is no child benefit.

Example 1. If there is no child benefit: $\varphi = 0$, then the balance condition [(9) below] is simply $\theta = \gamma$. The introduction of a pure basic income has no fertility effect, because the optimal fertility is independent of the net income z:

$$n_0 = n(w, 0, \theta) = \frac{\zeta}{\bar{\zeta}\pi}. \tag{6°}$$

Then the young worker's consumption is equal to $c_0 = z/\bar{\zeta}$, independently of the per-child raising cost π. We are especially interested in cases, where the fertility is below 1, therefore we shall assume $n_0 < 1$, i.e. $\pi > \zeta/\bar{\zeta}$.

Returning to the general case of $\varphi > 0$, we present the explicit solution of the wage-specific fertility.

Lemma 1. *In a(n unbalanced) transfer system with basic income γ and child benefit rate φ, the optimal fertility of a young worker with a narrow income $z = \hat{\theta} w + \gamma$ is the positive root of the quadratic equation*

$$E(n, w, \varphi, \gamma) = (2 + \zeta)\pi\varphi n^2 + \bar{\zeta}(\pi z - \varphi)n - \zeta z = 0, \quad \text{where} \quad \bar{\zeta} = 1 + \zeta, \tag{6}$$

namely

$$n(w, \varphi, \gamma) = \frac{-B + \sqrt{B^2 + 4AC}}{2A}, \tag{7}$$

where

$$A = (2 + \zeta)\pi\varphi, \qquad B = \bar{\zeta}(\pi z - \varphi) \quad \text{and} \quad C = \zeta z. \tag{8}$$

Though formulas (7)–(8) are helpful in the numerical calculations, their analytical use is very limited. In contrast, (5) is very useful, implying

Corollary. *The individual optimal fertility is an increasing function of the child benefit rate $\varphi > 0$, a decreasing function of the wage $w > 0$ and of the basic income $\gamma > 0$. The higher the wage, the weaker is the impact of the rise in the transfer rates φ and γ.*

Indeed, consider the optimality condition (5) and increase φ, then the declining curve shifts to the right, leading to a higher $n(w, \varphi, \gamma)$. The opposite holds for w and γ. The higher the w, the higher is z, thus the lower is the right hand side of (5).

Macrorelations

Having finished the individual analysis, we consider the whole population with a given wage distribution and assume that the average wage is $\mathbf{E}w = 1$, regardless of age. We have four overlapping generations, where N denotes the size of generation born in the current period and N_{-i} denotes that of last but i period, $i = 1, 2, 3$. Then the current and past average fertilities are equal $\nu = \mathbf{E}n = N/N_{-1}$ and $\nu_{-1} = \mathbf{E}n_{-1} = N_{-1}/N_{-2}$, respectively. Since pensioners are neglected, N_{-3} is irrelevant.

We shall now discuss the balance condition of the tax system in the current period. On average, the tax is the sum of the child benefit and the basic income, the former being the product of the child benefit rate and the number of children. Hence

$$(N_{-1} + N_{-2})\theta = \varphi N + (N_{-1} + N_{-2})\gamma. \tag{9}$$

Using definitions $N_{-1} = \nu_{-1}N_{-2}$ and $N = \nu\nu_{-1}N_{-2}$, (9) simplifies to the balance equation

$$(1 + \nu_{-1}^{-1})(\theta - \gamma) = \varphi\nu(\varphi, \gamma), \tag{9$'$}$$

where $\nu(\varphi, \gamma) = \mathbf{E}n(w, \varphi, \gamma)$ is the average current fertility as a function of the child benefit rate φ and the basic income γ, $\nu_{-1}^{-1} = 1/\nu_{-1}$ is sometimes suppressed.

Before presenting the involved analysis, we discuss another very simple special case.

Example 2. Assume homogeneous wages: $w \equiv 1$, and exclude basic income: $\gamma = 0$, hence $z = 1 - \theta$. To simplify the calculations, we are looking for the stationary case $n = 1$, when the balance condition (9$'$) reduces to $\varphi = 2\theta$. Substituting into (6) directly:

$$E(1, 1, 2\theta, 0) = 2(2 + \zeta)\pi\theta + \bar{\zeta}(\pi(1 - \theta) - 2\theta) - \zeta(1 - \theta) = 0.$$

By simple calculation, the corresponding tax rate is given by

$$\theta_1 = \frac{\bar{\zeta}\pi - \zeta}{2 + \zeta - (3 + \zeta)\pi} \in (0, 1) \quad \text{if} \quad \frac{\zeta}{1 + \zeta} < \pi < \frac{1 + \zeta}{2 + \zeta}.$$

We return now to the general problem. For a fixed tax rate θ, a child benefit rate φ and a past average fertility rate ν_{-1}, one has to substitute (7)–(8) into (9$'$), and solve the resulting implicit equation for $\gamma[\varphi, \nu_{-1}]$ called *balanced* basic income:

$$(1 + \nu_{-1}^{-1})(\theta - \gamma[\varphi, \nu_{-1}]) = \varphi\nu(\varphi, \gamma[\varphi, \nu_{-1}]). \tag{9$''$}$$

We are looking for the conditions guaranteeing the existence and uniqueness of the balanced income, moreover $\gamma[\varphi, \nu_{-1}]$ is decreasing in φ. Proofs are given in Appendix A.

Theorem 1. *Fix the past average fertility* ν_{-1} *and let* $\varphi_{\rm m}$ *be a positive real number such that*

$$(1+\nu_{-1}^{-1})\theta > \varphi\nu(\varphi, 0) \tag{10a}$$

and

$$-\varphi\nu'_\gamma(\varphi,\gamma) < 1+\nu_{-1}^{-1}, \qquad \gamma < \theta \tag{10b}$$

hold for $0 < \varphi < \varphi_{\rm m}$. *Then the balanced basic income* $\gamma[\varphi, \nu_{-1}]$ *exists, is unique and decreasing in* φ*; the average fertility* $\nu(\varphi, \gamma[\varphi, \nu_{-1}])$ *is increasing in* φ.

Remark. Condition (10a) is simple because the right hand side is an increasing function of φ. Condition (10b) is more complex, because the positive number on the left hand side looks much lower than the right hand side but we are not sure if this is an effective bound or not.

We can now formulate the fertility dynamics:

$$\nu[\varphi, \nu_{-1}] = \frac{(1+\nu_{-1}^{-1})(\theta - \gamma[\varphi, \nu_{-1}])}{\varphi}, \qquad \varphi > 0. \tag{9$'''$}$$

Note that this relation is essentially independent of the utility function, only the form of $\gamma[\varphi, \nu_{-1}]$ depends on the optimization framework.

Under certain (unexplored) conditions and for a given child benefit rate φ, there exists a unique (balanced) steady-state (average) fertility $\nu^{\rm o}$ which satisfies

$$\theta = \varphi\frac{\nu^{\rm o2}}{\nu^{\rm o}+1} + \gamma[\varphi, \nu^{\rm o}], \tag{9$^{\rm o}$}$$

where

$$\nu^{\rm o} = \mathbf{E}n[w, \varphi, \nu^{\rm o}].$$

We prove now the following theorem.

Theorem 2. *Assuming that there exists a unique steady-state fertility, it is an increasing function of the child benefit rate.*

Remark. For any feasible child benefit rate, we conjecture that the fertility dynamics (9$'''$) converges very fast to the steady state. The local convergence speed is the reciprocal of the contraction factor $|H'(\nu^{\rm o})|$. The fast convergence is obvious for $\varphi = 0$, when the dynamics steers fertility to the steady state n_0 in just one period, regardless of the initial fertility ν_{-1}. Our numerical experiments support this hypothesis for $\varphi > 0$ as well.

To choose the socially optimal child benefit system, the government maximizes a utilitarian social welfare function:

$$V[\varphi] = \mathbf{E}\left\{\log c[w, \varphi] + \zeta \log n[w, \varphi] + \log d[w, \varphi]\right\}. \tag{11}$$

We shall prove that some positive child benefit is socially useful, at least if the past average fertility rate was below 1. (For $\nu_{-1} \geq 1$, $\varphi^* = 0$.)

Theorem 3. *For any given past average fertility rate $\nu_{-1} < 1$, the socially optimal balanced child benefit rate is positive:* $\varphi^* > 0$.

As is usual, to compare the two systems—basic income combined with child benefits and pure basic income—from a welfare point of view, we introduce the following concept: the *relative efficiency* ε of the combined system with respect to the pure basic income is equal to that positive real number, multiplying the wages of the no-benefit system by it, the welfare is equal to that of the benefit system with the original wages. Adding an argument for the average wage ε in the social welfare function, the corresponding equation for efficiency is

$$V(1, \varphi) = V(\varepsilon, 0).$$

Due to the special structure of the utility and the social welfare functions,

$$V(\varepsilon, 0) = V(1, 0) + 2 \log \varepsilon, \quad \text{i.e.} \quad \varepsilon = \exp([V(1, \varphi) - V(1, 0)]/2). \tag{12}$$

Numerical illustrations: base run

To help the understanding of the steady state's and the welfare function's behavior, we shall display numerically the dependence of the optimal outcomes on the balanced transfer rates. In the base run, we have only two types, in the base run with earnings $w_L = 0.5$ and $w_H = 2$, with a common relative raising cost $\pi = 0.35$ and population shares $f_L = 2/3$, $f_H = 1 - f_L = 1/3$. We fix the value of the tax rate at $\theta = 0.3$ and the preference parameter at $\zeta = 0.4$. Note that this choice satisfies the condition set in Example 2: $0.286 < 0.35 < 0.59$. For $\varphi = 0$, $\gamma = \theta$ and $n_L = n_H = n_0$ hold (Example 1).

In Table 1 we display the dependence of the balanced basic income and the steady state fertility as a function of the child benefit rate. Note that in reality, the pure child benefit rate is very low, in Hungary, about 2% of the average total wage cost. But including hidden transfers like free schooling and health care, this rate can be much higher. As the benefit rate rises from 0 to 0.16 (close to the maximum), the balanced basic income drops from 0.3 to 0.225 and the steady-state fertility rises from $n_0 = 0.816$ to 1.051.

Table 1. *Child benefit rate and steady-state fertility*

Child benefit rate φ	Balanced basic income $\gamma[\varphi, \nu^o]$	Steady-state fertility ν^o
0	0.300	0.816
0.04	0.284	0.876
0.08	0.267	0.936
0.12	0.247	0.995
0.16	0.225	1.051

Remark. $w_L = 1/2$, $f_L = 2/3$, $\theta = 0.3$.

Moving from the steady state analysis to the dynamic one, we should pay attention to the influence of past average fertility. Creating Table 2 we fix the past average fertility rate, namely below 1: close to n_0, $\nu_{-1} = 0.8$. As the child benefit rate increases, the relative efficiency increases less and less and reaches the maximum around $\varphi^* = 0.06$. Increasing φ until it almost covers the raising cost of the lower paid type, the fertility of the lower paid increases much faster than the higher paid's: $n_L = 1.127 > 0.939 = n_H$. Note, however, that the social welfare slowly sinks, and from $\varphi = 0.12$ it drops below the no-benefit level.

Table 2. *Impact of the child benefit rate*

Child benefit rate φ	Balanced basic income $\gamma[\varphi, \nu_{-1}]$	Fertility: Low wage n_L	Fertility: High wage n_H	Fertility: Average ν	Relative efficiency ε
0.00	0.300	0.816	0.816	0.816	1.000
0.06	0.276	0.929	0.859	0.906	1.002
0.12	0.247	1.041	0.902	0.995	1.000
0.17	0.220	1.127	0.939	1.065	0.995

Remark. See Table 1, $\nu_{-1} = 0.8$.

3. Family tax deduction system

There are governments which are worried by the large transfers flowing from high-earner workers (families) to low-earner ones through child benefits. To mitigate this unwanted consequence, these governments replace child benefits by family tax deductions. (It is possible to model a partial replacement but it would unnecessarily complicate the analysis.) The essence of the family tax deduction is that only the higher wage types can fully use it: any positive excess transfer $e = \varphi n(w) - \theta w$ is eliminated.

The simplest formulation of the family tax deduction is as follows. Let $\psi > 0$ be the child tax deduction rate, i.e. having n children, amount ψn can be deducted from the proportional personal income tax θw, up to the maximum θw. To avoid absurd cases, we assume that the family tax deduction is always lower than or equal to the narrow raising cost: $\psi \le \pi z_{\mathrm{m}}$, where z_{m} corresponds to the minimal wage w_{m}. Let t_0 denote now the tax deducted: $t_0 = \min(\theta w, \psi n)$. Obviously, if the benefit is so low or the tax rate is so high that even the minimal wage earner's tax amount is higher than the family tax deduction, then the latter reduces to the child benefit. But this has already been covered in Section 3, therefore we assume that $\theta w_{\mathrm{m}} \le \psi n$.

By definition, type w's old-age consumption is equal to $d = z$, while its own young-age consumption is equal to

$$c = (z + t_0)(1 - \pi n) \tag{13}$$

We have now two domains in the parameter space $(w, \theta, \varphi, \gamma) \in \mathbf{R}_+^4$; *slack*, denoted by S: $\theta w_{\mathrm{S}} > \psi n_{\mathrm{S}}$ and *tight*, denoted by T: $\theta w_{\mathrm{T}} \le \psi n_{\mathrm{T}}$. (The status of the demarcation line $\theta w = \psi n$ is ambiguous.) Correspondingly, $t_{0S} = \psi n$ and $t_{0T} = \theta w$. Then (13) branches off into

$$c_{\mathrm{S}} = (z_{\mathrm{S}} + \psi n_{\mathrm{S}})(1 - \pi n_{\mathrm{S}}), \qquad z_{\mathrm{S}} = \hat{\theta} w_{\mathrm{S}} + \gamma \tag{13S}$$

and

$$c_{\mathrm{T}} = (w_{\mathrm{T}} + \gamma)(1 - \pi n_{\mathrm{T}}), \qquad z_{\mathrm{T}} = \hat{\theta} w_{\mathrm{T}} + \gamma. \tag{13T}$$

Then there are two separate regimes with their own fertility optima. Lemma 1 provides n_{S} for ψ replacing φ in (7)–(8), while n_{T} for $\varphi = 0$ and $\theta = 0$, i.e. (6°). It can be shown that $n_{\mathrm{S}}(w, \psi, \gamma) > n_{\mathrm{T}}$. It is especially disturbing that the transition from S into T is discontinuous: the optimal transfer drops a lot due to a minor tax rate rise!

To formulate the new balance condition, we repeat the argument leading to (9). But now we take into account the partition along S–T, which depends on the parameter vector (w, ψ, γ). For convenience, we assume that the relevant functions $n_{\mathrm{S}}(w, \psi, \gamma)$ and $n_{\mathrm{T}}(w, \psi, \gamma)$ are also defined outside their natural domains, being equal to zero outside their proper domains. Subindexes S and T refer to these restricted expectations. The reformulated balance equation (cf. (9′)) is as follows:

$$(1 + \nu_{-1}^{-1})(\theta - \gamma) = \psi \mathbf{E}_{\mathrm{S}} n_{\mathrm{S}}(w, \psi, \gamma) + \theta \mathbf{E}_{\mathrm{T}} w. \tag{14}$$

Our social welfare function remains basically the same as above, only ψ replaces φ.

Due to its simplicity, it is worth discussing the two-type case of family tax deduction.

Example 3. In the two-type case, the low-wage type is tight, the high-wage type is slack. Furthermore, at the social optimum, the low wage type's family tax deduction is equal to her tax: $\psi^* n_L = \theta w_L$. Inserting (6°), our optimality condition becomes

$$\psi^* = \frac{\bar{\zeta} \pi \theta w_L}{\zeta}. \tag{15}$$

Substituting (15) into (14) and using (7)–(8) yield an equation for $\gamma[\psi^*, \nu_{-1}]$.

It is easy to grasp that in general the optimal family tax deduction system is far from being socially optimal. Due to the elimination of the excess transfer, the low-wage type's fertility is as low as n_0 in (6°) and the corresponding net income is only $y_L = w_L + \gamma$, which is lower than in the pure basic income system: $y_L^{\mathrm{o}} = \hat{\theta} w_L + \theta$. The high-wage type's fertility is higher than n_0 but the additional resource brings less gain in her consumption and child welfare than the loss is in the low-wage type's welfare.

We continue the numerical illustrations. For $\theta = 0.3$, the optimal tax deduction rate is $\psi^* = 0.184$ and the corresponding basic income $\gamma^* = 0.230$. The fertility rates are respectively $n_L = 0.816$, $n_H = 0.948$ and $\nu = 0.860$. The relative efficiency is $\varepsilon = 1.002$, just the same as the child benefit system's.

4. Check of robustness

In this Section we shall check the robustness of our numerical results. We also add a new indicator to be called the double weighted fertility: $\nu_w = \mathbf{E}(wn(w))$, where the wage dependent fertilities $n(w)$ are also weighted by the parents earnings, approximating the average quality of children.

Creating Tables 3a and b, the fertility and the consumption outcomes of the socially optimal child benefit system reported separately. We shall raise the low wage from 0.5 to 0.75 and independently, diminish the low earners' frequency from 2/3 to 1/3. Note that for given frequencies, the high wage changes according to $w_H = (1 - f_L w_L)/f_H$. On the other hand, if the wages are given, $w_L < 1 < w_H$, then the corresponding frequency is determined as $f_L = (w_H - 1)/(w_H - w_L)$. To help the reader, we shall display f_L in Tables 3a–4a, and w_H in Tables 3b–4b.

It may be surprising that the socially optimal benefit rate and the balanced basic income are insensitive to the wage and frequency distribution, they are around 0.07 and 0.27, respectively. The same invariance applies to the average fertility (around 0.91), the wage-weighted fertility (around 0.89) and the relative efficiency (around 1.003–1.004). We shall see in Appendix B that in the approximation, this invariance of average fertility is exact. The efficiency gain may seem to be modest, it is usual in such welfare calculations. The only noticeable change occurs within the fertilities: the lower wage earners' fertility varies between 0.92 to 0.95, while the higher wage earners' fertility varies between 0.86 and 0.90.

Table 3a. *Optimal child benefits for varying wages and frequencies: fertilities*

Low wage w_L	LE frequency f_L	Child benefits rate φ^*	Balanced basic income $\gamma[\varphi^*, \nu_{-1}]$	Fertility: Low wage n_L	Fertility: High wage n_H	Fertility: Average ν	Relative efficiency ε	Double weighted fertility ν_w
0.50	0.667	0.065	0.274	0.938	0.863	0.913	1.002	0.888
	0.500	0.068	0.273	0.944	0.877	0.911	1.002	0.894
	0.333	0.071	0.271	0.950	0.890	0.910	1.002	0.900
0.75	0.667	0.070	0.272	0.920	0.879	0.906	1.002	0.899
	0.500	0.071	0.272	0.921	0.890	0.905	1.002	0.902
	0.333	0.072	0.271	0.923	0.897	0.905	1.002	0.903

Remark. LE = low earner, $\nu_{-1} = 0.8$, $\theta = 0.3$

Turning to the details of consumption, Table 3b shows that as w_L and w_H change, the low earners' consumption at young and old ages increase. Note that the high earners' old-age incomes and the lifetime utilities decrease when f_L drops or w_L rises, since w_H drops.

Table 3b. *Optimal child benefits for varying wages and frequencies: consumption*

Low earners'			High earners'		
	younger	older		younger	older
wage	consumption		wage	consumption	
w_L	c_L	d_L	w_H	c_H	d_H
0.5	0.460	0.624	2.000	1.208	1.674
		0.623	1.500	0.958	1.323
		0.621	1.250	0.833	1.146
0.75	0.584	0.797	1.500	0.958	1.322
		0.797	1.250	0.833	1.147
		0.796	1.125	0.771	1.059

Remark. Table 3a

Turning to the socially optimal family tax deduction system, Tables 4a and b are more varied. By (15), the optimal tax deduction rate ψ^* rises with w_L, from 0.184 to 0.276. The low-earners' fertility rate now stagnates at $n_L = n_0$, while the high earners' fertility rises from $n_H = 0.948$ (at $w_L = 0.5$ and $f_L = 2/3$) to 1.117 (at $w_L = 0.75$ and $f_L = 1/3$). The average fertility also increases, but note that at transition from $w_L = 0.5$ and $f_L = 1/3$ to $w_L = 0.75$ and $f_L = 2/3$, the average fertility drops from 0.94 to 0.9. The relative efficiency is sinking from 1.004 to 0.992.

Table 4a. *Optimal family tax deductions for varying wages and frequency: fert*

Low wage w_L	LE's frequency f_L	Tax deduction rate ψ^*	Balanced basic income $\gamma[\psi^*, \nu_{-1}]$	Fertility Low wage n_L	Fertility High wage n_H	Average ν	Relative efficiency ε	Double weighted fertility ν_w
0.50	0.667	0.184	0.230	0.816	0.948	0.860	1.002	0.904
	0.500		0.227		0.982	0.899	1.000	0.941
	0.333		0.223		1.007	0.943	0.999	0.975
0.75	0.667	0.276	0.190	0.816	1.061	0.898	0.999	0.939
	0.500		0.183		1.096	0.956	0.995	0.991
	0.333		0.175		1.117	1.017	0.992	1.042

Turning to the details of consumption, a similar picture emerges as in Table 3b but the dropping frequency f_L slightly decreases the low earners' consumption at both ages. Note that the younger age consumption dramatically increases, while the old-age consumption dramatically decreases with respect to the benefit system. Their sum changes from 1.08 to 1.1 but is less diverse (in the basic run). Turning to high earners, the total consumption decreases from 2.882 to 2.835 (also in the basic run) but the much higher fertility compensates for this drop. We must admit that it is not easy to understand the welfare differences between the two systems.

Table 4b. *Optimal family tax deductions for varying wages and frequencies: cons*

Low earners'			High earners'		
	younger	older		younger	older
wage	consumption		wage	consumption	
w_L	c_L	d_L	w_H	c_H	d_H
0.5	0.521	0.580	2.000	1.205	1.630
	0.519	0.577	1.500	0.956	1.277
	0.516	0.573	1.250	0.831	1.098
0.75	0.671	0.715	1.500	0.963	1.240
	0.666	0.708	1.250	0.838	1.058
	0.661	0.700	1.125	0.774	0.963

Until now, we have fixed the tax rate at $\theta = 0.3$. If we change its value between 0 and 0.5, and compare the two systems, we get Tables 5 and 6.

Starting with the child benefit systems, in Table 4 we see the same invariance as before in Table 3. The difference between the tax rate and the basic income is roughly constant, approximately 0.024. (Note that in the approximation used in Appendix B, this difference is exactly constant.) The relative efficiency is above that of the pure tax system by 0.2%. A surprising phenomenon occurs in the interval of unrealistically low the tax rates [0, 0.06]: while θ rises, the low earners' fertility jumps from 0.816 to 0.95, the high earners' fertility n_H rises from 0.816 to 0.854; the optimal child benefit rate reaches 0.06 and the relative efficiency stabilizes around 1.004.

Table 5. *The impact of the tax rate on optimal benefits*

	Child	Balanced		Fertility			Double
Tax	benefit	basic	Low	High		Relative	weighted
rate	rate	income	wage		Average	efficiency	fertility
θ	φ^*	$\gamma[\varphi^*, \nu_{-1}]$	n_L	n_H	ν	ε	ν_w
0.0	0.000	0.000	0.816	0.816	0.816	1.000	0.816
0.1	0.059	0.076	0.947	0.854	0.916	1.002	0.885
0.2	0.062	0.175	0.943	0.858	0.914	1.002	0.886
0.3	0.065	0.274	0.938	0.863	0.913	1.002	0.888
0.4	0.067	0.373	0.933	0.867	0.911	1.002	0.889
0.5	0.068	0.473	0.927	0.871	0.908	1.002	0.890

Remark. $w_L = 2/3$ and $f_L = 2/3$.

Continuing with the tax deduction system, we see that the difference $\theta - \psi^*$ increases from 0.023 to 0.122. In $0.1 \le \theta < 0.3$, its relative efficiency is higher; in $0.3 < \theta \le 0.5$, its relative efficiency is lower than the benefit system's. Even if the consumption data were shown as in Tables 3b–4b, it would be difficult to understand the relation between two sums of six terms.

Table 6. *The impact of the tax rate on optimal deductions*

Tax rate θ	Tax deduction rate ψ^*	Balanced basic income $\gamma[\psi^*, \nu_{-1}]$	Fertility Low wage n_L	Fertility High wage n_H	Fertility Average ν	Relative efficiency ε	Double weighted fertility ν_w
0.1	0.061	0.077	0.816	0.855	0.829	1.003	0.842
0.2	0.123	0.154		0.899	0.844	1.003	0.871
0.3	0.184	0.230		0.948	0.860	1.002	0.904
0.4	0.245	0.304		1.002	0.878	1.000	0.940
0.5	0.306	0.378		1.061	0.898	0.996	0.980

Remark. see Table 5

Finally we illustrate the impact of the past average fertility on the two optima, fixing the tax rate at $\theta = 0.3$. Table 7 displays the important outcomes while ν_{-1} changes between 0.7 and 0.9. The most important impact concerns the optimal child benefit rate and the basic income: the first drops from 0.118 to 0.028, while the second rises from 0.252 to 0.289. As a result, all the four fertilities decrease, namely the average one from 0.991 to 0.858. Here are the details.

Table 7. *The impact of the past average fertility on optimal benefits*

Past average fertility ν_{-1}	Child benefit rate φ^*	Balanced basic income $\gamma[\psi^*, \nu_{-1}]$	Fertility Low wage n_L	Fertility High wage n_H	Fertility Average ν	Relative efficiency ε	Double weighted fertility ν_w
0.70	0.118	0.252	1.036	0.901	0.991	1.005	0.946
0.75	0.088	0.265	0.981	0.879	0.947	1.003	0.913
0.80	0.065	0.274	0.938	0.863	0.913	1.002	0.888
0.85	0.045	0.282	0.901	0.848	0.883	1.001	0.866
0.90	0.028	0.289	0.868	0.836	0.858	1.000	0.847

Remark. See Table 1.

Repeating this calculation for the family tax deduction, the change is mainly limited to basic income, which drops from 0.235 to 0.225. As a consequence, the relative efficiency drops from 1.009 to 0.996. Note that for low past average fertility, this is much higher than the child benefit optimum, and for high past average fertility, it is much lower.

5. Conclusions

In our very simple model, we studied the interaction of the personal income tax and fertility-related transfers. We studied analytically the socially optimal child benefit and the family tax deduction systems. At least in our arbitrary numerical examples, the socially optimal child benefit system's efficiency is constant, while the family tax deduction system's efficiency sensitively depends on the parameter values of the tax rate and the wage distribution. Counterintuitively, on the one hand, if the tax rate is high and the wage distribution is modestly unequal, than the benefit system is more efficient than the deduction system is. Then even the pure basic income system may overtake the deduction system. On the other hand, if the tax rate is low and the wage distribution is very unequal, than the opposite holds. It is to be underlined that in practice, the family tax deduction system may be very far from this optimum, when the personal income tax rate is so low that only the high earners can use the deduction sufficiently.

We warn the reader on the limits of the model. We used the simplest utility function pair, two logarithmic functions. Even at the modest generalization into CRRA (see, e.g. Greenwood, Sheshadri and Vandenbroucke, 2005), the independence of the fertility of the wage in Example 1 (no child benefit) disappears, therefore the saving of Theorems 1 and 2 requires further nontrivial assumptions. The neglect of the negative impact of taxation on labor supply and tax reporting further weakens the force of our numerical examples. The inclusion of labor disutility and flexible labor supply or tax morale and underreporting would make the model more realistic and determine the optimal tax rate, would more fully highlight the differences between the two transfer systems. But these modifications would further complicate the analysis, therefore we have not used them here. The heterogeneity of the relative raising cost also deserves an examination.

Appendix A: Proofs

Proof of Theorem 1

Define the balance of the transfer system as

$$D(\varphi,\gamma,\nu_{-1}) = (1+\nu_{-1}^{-1})(\theta-\gamma) - \varphi\nu(\varphi,\gamma)$$

and consider it as a function of the second variable γ. Since

$$D(\varphi,0,\nu_{-1}) = (1+\nu_{-1}^{-1})\theta - \varphi\nu(\varphi,0) > 0 > D(\varphi,\theta,\nu_{-1}) = -\varphi\nu(\varphi,\theta),$$

therefore, by Bolzano-theorem, there exists at least one root for $D(\varphi,\gamma,\nu_{-1}) = 0$.

To prove uniqueness, note that by our second assumption,

$$D'_\gamma(\varphi,\gamma,\nu_{-1}) = -(1+\nu_{-1}^{-1}) - \varphi\nu'_\gamma(\varphi,\gamma) < 0 \quad \text{for} \quad 0 < \varphi < \varphi_{\mathrm{m}}.$$

We shall also need

$$D'_\varphi(\varphi,\gamma,\nu_{-1}) = -\nu(\varphi,\gamma) - \varphi\nu'_\varphi(\varphi,\gamma) < 0$$

(see Corollary for $\nu'_\varphi > 0$).

Using the implicit function theorem,

$$\gamma'[\varphi] = -\frac{D'_\varphi}{D'_\gamma} < 0.$$

Turning to the balanced fertility–benefit rate schedule, take the total derivative of $\nu(\varphi, \gamma[\varphi, \nu_{-1}], \nu_{-1})$ by φ:

$$\frac{d}{d\varphi}\nu(\varphi, \gamma[\varphi, \nu_{-1}]) = \nu'_\varphi + \nu'_\gamma \gamma'_\varphi[\varphi, \nu_{-1}],$$

where $\nu'_\varphi > 0 > \nu'_\gamma$, thus the total derivative is positive. ∎

Proof of Theorem 2

We shall rely on the implicit function theorem again. Dropping the superscript o, introduce notation

$$G(\varphi, \nu) = \varphi\frac{\nu^2}{\nu+1} + \gamma[\varphi, \nu].$$

Taking the partial derivatives

$$G'_\varphi = \frac{\nu^2}{\nu+1} + \gamma'_\varphi[\varphi, \nu] \quad \text{and} \quad G'_\nu = \varphi\frac{\nu^2+2\nu}{(\nu+1)^2} + \gamma'_\nu[\varphi, \nu].$$

By Theorem 1,

$$\gamma'_\varphi[\varphi, \nu] < \frac{\nu^2}{\nu^2+1}.$$

We shall demonstrate that $\gamma'_\nu[\varphi, \nu] > 0$.

Returning to $D(\varphi, \gamma, \nu_{-1})$, $\gamma'_\nu[\varphi, \nu] = -D'_\gamma/D'_{\nu_{-1}}$, where $D_\gamma < 0$ by assumption. A simple calculation yields

$$D'_{\nu_{-1}} = (\theta - \gamma) - \varphi\nu(\varphi, \gamma).$$

Comparing it to $D = 0$, yields $D'_{\nu_{-1}} > 0$. Therefore $G'_\varphi < 0 < G'_\nu$, i.e. $\nu'(\varphi) = -G'_\varphi/G'_\nu > 0$. ∎

Proof of Theorem 3

The existence of the social optima is obvious. The basic idea is borrowed from the well-known proof of the envelope-theorem. We shall show $V'[0] > 0$. Taking the derivative of V in (11) with respect to φ and using (1b) yield

$$V'[\varphi] = \mathbf{E}\frac{z'_\varphi[w,\varphi] + n[w,\varphi] + \varphi n'_\varphi[w,\varphi]}{z[w,\varphi] + \varphi n[w,\varphi]}$$

$$-\mathbf{E}\frac{\pi n'_\varphi[w,\varphi]}{1-\pi n[w,\varphi]} + \zeta\mathbf{E}\frac{n'_\varphi[w,\varphi]}{n[w,\varphi]} + \mathbf{E}\frac{z'_\varphi[w,\varphi]}{z[w,\varphi]}.$$

Using the individual optimality condition (5), multiplying it by $n'_\varphi[w,\varphi]$ and applying $z'_\varphi[w,\varphi] = \gamma'[\varphi]$ [(1a)], we obtain

$$V'[\varphi] = \mathbf{E}\frac{\gamma'[\varphi] + n[w,\varphi]}{z[w,\varphi] + \varphi n[w,\varphi]} + \mathbf{E}\frac{\gamma'[\varphi]}{z[w,\varphi]}.$$

For $\varphi = 0$, $n[w,0] = n_0$, the common denominator is $z[w,0]$ and the wage-dependent numerator of $V'(0)$ is a constant:

$$2\gamma'[0] + n_0 = n_0 - 2\frac{\nu_{-1}}{\nu_{-1}+1}n_0 = \frac{1-\nu_{-1}}{\nu_{-1}+1}n_0.$$

Therefore $V'[0] > 0$ if and only if $\nu_{-1} < 1$. ∎

Appendix B: Approximation of raising costs

In this Appendix, we shall discuss an approximation of the raising costs, used in the literature. On the one hand, this simplification opens more room for analytical calculations and enables us to prove directly some of our theorems above. On the other hand, it distorts the analysis, causing qualitative and quantitative errors.

Earlier papers (GLM and Simonovits, 2013, 2014) made the raising costs independent of the net income: $(\pi - \varphi)n$, implying young workers' consumption

$$c = z - (\pi - \varphi)n. \tag{B.1}$$

Due to (B.1),

$$u(n) = \log(z - (\pi - \varphi)n) + \zeta \log n + \log z, \tag{B.2}$$

therefore the optimal wage-specific fertility is

$$n(w,\varphi,\gamma) = \frac{\zeta z}{\bar{\zeta}(\pi - \varphi)}. \tag{B.3}$$

Note that contrary to Corollary to Lemma 1, the unbalanced fertility is an increasing rather than decreasing function of the wage and the basic income! Taking the expectations on $n(w,\varphi,\gamma)$ in (B.3), and using $z = \hat{\theta}w + \gamma$, i.e. $\mathbf{E}z = \hat{\theta} + \gamma$, i.e. the current average fertility is given by

$$\nu(\varphi,\gamma) = \frac{\zeta(\hat{\theta} + \gamma)}{\bar{\zeta}(\pi - \varphi)}. \tag{B.4}$$

Making a short detour, we remark that the aggregate fertility is now independent of the wage distribution!

In a better approximation, the per child raising costs are not independent of but proportional to the narrow net income, implying

$$c = z - (\pi z - \varphi)n. \tag{B.1'}$$

This modification would preserve the linearity but the resulting

$$n(w,\varphi,\gamma)=\frac{\zeta z}{\bar{\zeta}(\pi z-\varphi)} \qquad (B.3')$$

would make the average fertility dependent on the wage distribution, preventing simple aggregation.

Rather than determining the balanced basic income, we introduce the difference between the tax rate and the basic income: $\lambda=\theta-\gamma$ (i.e. $\gamma=\theta-\lambda$). This is the part of the tax which finances the child benefits, shortly: *earmarked child tax rate.* To shorten (B.4), we introduce notation $\omega=\zeta/[\bar{\zeta}(\pi-\varphi)]$, yielding

$$\nu(\varphi,\theta-\lambda)=(1-\lambda)\omega. \qquad (B.5)$$

(Note that for $\varphi=0$, $\omega=n_0$.) Substitute (B.5) into the balance equation (9′):

$$(1+\nu_{-1}^{-1})\lambda=\omega\varphi(1-\lambda),$$

hence the balanced earmarked child tax rate is equal to

$$\lambda[\varphi,\nu_{-1}]=\frac{\omega\varphi}{1+\omega\varphi+\nu_{-1}^{-1}}.$$

Note that the balanced earmarked tax rate is independent of the tax rate and is an increasing function of the child benefit rate. Therefore the balanced basic income is a decreasing function of the child benefit rate and any increase in the tax rate increases the income by the same quantity.

Returning to (B.5) yields a simple fertility dynamics:

$$\nu=\frac{\omega(1+\nu_{-1}^{-1})}{1+\omega\varphi+\nu_{-1}^{-1}}$$

i.e.

$$\nu=H(\nu_{-1})=\frac{\omega(1+\nu_{-1})}{1+(1+\omega\varphi)\nu_{-1}}. \qquad (B.6)$$

To obtain the steady state, we substitute $\nu=\nu_{-1}$ into (B.6). The resulting quadratic equation

$$(1+\omega\varphi)\nu^2+(\omega-1)\nu-\omega=0$$

yields the positive root

$$\nu^{\rm o}=\frac{\omega-1+\sqrt{(\omega+1)^2+4\omega^2\varphi}}{2(1+\omega\varphi)}. \qquad (B.7)$$

Finally we could check the local stability under (B.6). As is well-known, the dynamics is locally stable if $|H'(\nu^{\rm o})|<1$ holds. A simple calculation yields the condition of local stability:

$$|H'(\nu^{\rm o})|=\frac{\omega^2\varphi}{[1+(1+\omega+\varphi)\nu^{\rm o}]^2}<1. \qquad (B.8)$$

By geometric reasoning it could be proved that local stability here implies global stability.

In summary, we proved

Theorem B.1. *In approximation (B.1), the balanced steady-state average fertility is determined by (B.7) and it is locally (and globally) stable if (B.8) holds.*

We have tried to give a quite general and plausible condition for (B.8) to hold but we must be satisfied with numerical simulations presented in Table B.1. We display the dependence of the approximated steady state fertility as a function of the child benefit rate. As the benefit rate rises from 0 to 0.08, the balanced basic income drops from 0.3 to 0.259 (the exact value is equal to 0.267 in Table 1) and the steady-state fertility rises from $n_0 = 0.816$ to 1.015 (the exact value is equal to 0.936). The approximation is hardly acceptable.

Table B.1. *Child benefit rate and steady-state fertility: approximation*

Child benefit rate φ	Balanced basic income $\gamma[\varphi, \nu^o]$	Steady-state fertility ν^o	Contraction factor $\lvert H'(\nu^o)\rvert$
0	0.300	0.816	0
0.04	0.283	0.906	0.009
0.08	0.259	1.015	0.020
0.12	0.226	1.150	0.034
0.16	0.180	1.323	0.052

Remark. See Table 1.

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